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Efeitos de termos de ordem superior e acoplamentos não-mínimos sobre a gravitação de Einstein-Hilbert

Orientador: Prof. Dr. Manoel Messias Ferreira Junior

São Luís – MA
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RESUMO

Esta dissertação tem por objetivo revisar aspectos da gravitação de Einstein, focando nas técnicas de cálculo do propagador de Feynman do gráviton na teoria de Einstein-Hilbert e em modelos gravitacionais modificados. Para esse fim, apresentamos e usamos o formalismo de projetores tensoriais, adequados para a realidade do campo gravitacional (campo de spin-2), na famosa base de Barnes-Rivers. Frisamos que para cada modelo de gravidade considerado existe um conjunto distinto de operadores que formam uma álgebra fechada, e que o formalismo de projetores tensoriais só funciona quando consideramos o conjunto adequado para cada cenário específico. Tal formalismo, além de muito elegante, é bastante poderoso por permitir calcular o propagador do gráviton em teorias gravitacionais diversas. De posse do propagador, obtemos as relações de dispersão, que permitem acessar informações atinentes a propagação de sinais e sobre as excitações da teoria. Neste trabalho, revisamos o cálculo do propagador do gráviton em quatro teorias distintas. A primeira delas é a gravitação de Einstein-Hilbert não-massiva, onde consideramos a densidade lagrangiana apenas composta pelo escalar de Ricci (R). A segunda consiste numa versão estendida da teoria gravitacional de Einstein-Hilbert, modificada por termos quadráticos, ou seja, além do escalar de Ricci, a lagrangiana desse modelo conta com termos do tipo R^2 e $R_{\mu\nu}R^{\mu\nu}$. A terceira é a gravitação de Einstein-Chern-Simons, que é uma teoria para gravidade em $(2+1)$ dimensões. Neste modelo, os operadores que constituem a base de Barnes-Rivers não formam um conjunto fechado, necessitando de uma versão estendida para a obtenção do propagador. Por fim, abordamos uma teoria de gravitação constituída pelo acoplamento entre o campo bumblebee (b^μ) e a lagrangiana de Einstein-Hilbert. Aqui também se faz necessária uma extensão da base de Barnes-Rivers, através de projetores que contenham o campo bumblebee, que é o violador da simetria de Lorentz.

Palavras-chave: Einstein-Hilbert. Teorias gravitacionais. Propagador do gráviton. Barnes-Rivers. Modelo bumblebee. Violação da simetria de Lorentz.

ABSTRACT

This master thesis has the goal to revise some aspects of the Einstein's gravitation, focused on techniques of compute the graviton's Feynman propagator in Einstein-Hilbert theory and modified models of gravity. For this purpose, we show the tensorial projectors formalism suitable for the gravitational field case (spin-2 field) in the famous Barnes-Rivers basis. We emphasize that for each gravity model considered exist a distinct set of operators that form a closed algebra, and that the formalism of tensorial projectors only work when we consider the suitable set for each specific case. Such formalism, besides being very elegant, is quite powerful because it allows calculating the graviton's propagator in different gravitational theories. In possession of the propagator, we obtain the dispersion relations, which allow access to information related to the propagation of signals and about the excitations of the theory. In this work, we review the algorithm to compute the graviton's propagator in four different theories. The first of these is the non-massive Einstein-Hilbert gravitation, where we consider the Lagrangian density only composed by the Ricci scalar (R). The second one consists of an extended version of the Einstein-Hilbert gravitational theory, modified by quadratic terms, that is, in addition to the Ricci scalar, the Lagrangian of this model has terms of the type R^2 and $R_{\mu\nu}R^{\mu\nu}$. The third is the gravitation of Einstein-Chern-Simons, which is a theory for gravity in $(2+1)$ dimensions. In this model, the operators that compose the Barnes-Rivers basis do not form a closed set, requiring an extended version to obtain the propagator. Finally, we approach a theory of gravitation constituted by the coupling between the bumblebee field (b^μ) and the Einstein-Hilbert lagrangian. An extension of the Barnes-Rivers basis is also necessary here, through the projectors that contain the bumblebee field, which is the violator of Lorentz's symmetry.

Keywords: Einstein-Hilbert. Gravitational theories. Graviton's propagator. Barnes-Rivers. Bumblebee model. Lorentz's symmetry violation.

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SUMÁRIO

Introdução

Este trabalho aborda de maneira pormenorizada um procedimento bastante trabalhoso da física teórica: o cálculo do propagador de Feynman em teorias gravitacionais, que corresponde ao propagador do gráviton. Mesmo no cenário mais simples, que condiz à gravitação de Einstein-Hilbert (E-H), o procedimento exige um trato e habilidade com uma base de projetores especialmente construídos para a estutura de uma teoria de spin-2. A situação vai se tornando mais complexa à medida que novos termos vão sendo adicionados à ação de E-H. O fato de o campo gravitacional ser descrito por uma teoria de spin 2, exigindo ser representado por um tensor com dois índices, configura uma diferença substancial em relação ao cenário do campo de Maxwell (spin-1). Nessa dissertação, apresentaremos o procedimento de cálculo baseado no método dos projetores tensoriais, o mesmo usado com sucesso para o cálculo do propagador em teorias eletromagnéticas. Iniciamos, portanto, revisando o método de cálculo do propagador do campo eletromagnético dentro da teoria de Maxwell-Carrol-Field-Jackiw. A fim de familiarizar o leitor com o algoritmo a ser empregado no cálculo dos propagadores, apresentamos uma abordagem inicial considerando a teoria eletromagnética de Maxwell-Carrol-Field-Jackiw. Por possuir uma lagrangiana não-elementar (para uma teoria de spin-1), dotada de um campo de fundo que viola a simetria de Lorentz, a teoria de MCFJ torna-se um excelente ponto de partida para ilustração.

Para tratar o caso gravitacional, apresentamos a base de Barnes-Rivers, cujos operadores são combinações linearmente independente dos projetores longitudinal e transversal do campo eletromagnético em $(3+1)$ dimensões. Esta base será utilizada para expressar os operadores bilineares e o operador identidade, assim a álgebra dos projetores longitudinal e transversal nos permite construir a álgebra dos operadores de Barnes-Rivers. Ao mesmo tempo em que estes projetores ajudam no cálculo do propagador por um lado, tornam a tarefa mais trabalhosa por outro. Isso acontece porque alguns dos operadores da base de Barnes-Rivers possuem uma extensão significativa e, a depender dos operadores envolvidos na contração, surgem diversos cálculos a serem realizados, o que exige um nível de atenção considerável por parte de quem lida com a tarefa manualmente.

Na álgebra matricial, é sabido que "o produto de uma matriz por sua inversa é igual a matriz identidade". No cálculo do propagador de uma teoria de calibre, pragmaticamente do ponto de vista operacional, o produto da matriz é trocado por palavra produto por contração tensorial e matriz por operador bilinear. Isso se deve ao fato de o propagador de Feynman ser o inverso de um operador tensorial que "quadra" a lagrangiana em questão. Esse operador inverso pode ser proposto como uma combinação linear dos operadores de Barnes-Rivers. O procedimento consiste então em determinar os coeficientes dessa combinação linear e escrever adequadamente o operador identidade. Em álgebra matricial, a matriz identidade pode ser representada pelo delta de Kronecker que analogamente representa a métrica de um espaço euclidiano. Isso permite a concluir que em teorias de spin 1, o operador identidade pode ser escrito em termos do delta de Kronecker 4-dimensional, podendo ser representado por meio do tensor métrico. No entanto, em teorias gravitacionais, a métrica não pode assumir o papel de operador identidade por possuir um total de índices tensoriais (apenas dois) incompatível com o propagador para teorias de spin-2 (que possui quatro índices tensoriais). Uma estrutura em conformidade com essa condição, pode ser obtida pelo produto de tensores métricos que contenha um total de quatro índices tensoriais livres. Entretanto, não é qualquer configuração envolvendo o tensor métrico que cumpre a função de operador identidade. Tal arranjo deve ser compatível com a propriedade de neutralidade operacional, ou seja, sua aplicação em qualquer operador é sem efeito.

A presente dissertação se propõe a estudar o cálculo do propagador de Feynman em diferentes teorias gravitacionais. A primeira, e mais simples de todas, é a teoria de gravitação de Einstein-Hilbert, que corresponde ao modelo de gráviton não-massivo. Iniciamos o cálculo do propagador do gráviton não-massivo, levando em conta que a lagrangiana deste modelo é a de Einstein-Hilbert, composta apenas pelo escalar de Ricci. O propagador neste cenário é bem simples, obtemos apenas um polo que fornce a relação de dispersão de ondas planas, resultado típico de propagação não-massiva.

O segundo modelo abordado é o da gravitação de E-H modificada por termos quadráticos do tipo R^2 e $R_{\mu\nu}R^{\mu\nu}$. Nesta ocasião, o propagador possui polos massivos que surgem dos coeficientes de acoplamento presentes nos termos quadráticos. Salientamos que um modo massivo não indica necessariamente violação da causalidade, ou seja, podemos encontrar modos massivos causais em uma determinada teoria. Infelizmente, não é este o caso da gravitação quadrática, verificamos que um dos modos de propagação obtidos possui velocidade superluminal ($v > c$).

Seguindo com o cálculo do propagador em modelos de gravidade alternativos, como a gravitação de Einstein-Chern-Simons [15]. Embora essa teoria não contemple acoplamentos com campos de fundo fixos, o propagador obtido é modificado pela alteração na estrutra do espaço-tempo, promovida pela inserção do termo gravitacional de Chern-Simons. Assim como ocorre na teoria eletromagnética de Chern-Simons, nos deparamos com o surgimento de um termo proporcional ao símbolo de Levi-Civita, que aponta para a necessidade de extensão da base de Barnes-Rivers.

Por último, fazemos uma revisão da ref. [22] que consiste na obtenção do propagador do gráviton em uma teoria de campo com um setor que sofre violação da simetria d Lorentz. Nesse modelo, levamos em conta que um acoplamento entre o escalar de curvatura e o campo bumblebee (B^μ), violador da simetria de Lorentz numa escala de altas energias (escala de Planck), possua efeitos que possam ser observados em experimentos realizáveis em escalas de baixas energias . A presença do campo de fundo altera drasticamente a composição dos operadores que formam uma álgebra fechada, assim a base de Barnes-Rivers precisa ser estendida. A necessidade de extensão da base de Barnes-Rivers aparece no momento em que a lagrangiana que carrega os termos de VL é introduzida, pois surgem estruturas novas que envolvem combinações entre os projetores transversal e longitudinal com o campo bumblebee ($b_\mu\theta_{\alpha\beta}$ e $b_\mu\omega_{\alpha\beta}$). A extensão da base de Barnes-Rivers para este cenário foi apresentada em [38], a ideia por trás de sua execução é apresentada a seguir. Definimos os termos "estranhos" como novos operadores simetrizados, calculamos as contrações entre estes com o restante da base e observamos se a estrutura do resultado pode ou não ser escrita em termos das já previamente estabelecidas. Caso não seja possível, adicionamos os termos diferentes como novos operadores de spin e repetimos o processo até não haver mais o surgimento de termos estranhos . Não é objetivo deste trabalho obter a extensão da base de Barnes-Rivers no cenário de gravitação modificada por termos de violação da simetria de Lorentz. Entretanto, faremos o cálculo do seu propagador, analisando a sua estrutura de polos e compatibilidade com o princípio da causalidade.

CAPÍTULO 1

Cálculo do propagador para teorias de campo de spin 1

O algoritmo que aqui será empregado para o cálculo de propagadores pode parecer bem complicado para iniciantes. Tendo isso em vista, iremos fazer uma breve revisão deste procedimento em uma teoria de spin 1 por ser mais simples. O modelo que usaremos para apresentar as nuances do cálculo do propagador será o de Maxwell-Carrol-Field-Jackiw (MFCJ) que é um modelo que possui um campo de fundo (*background*) violador da simetria de Lorentz. Os primeiros estudos feitos sobre este modelo foram realizados por Sean M. Carrol, George B. Field e Roman Jackiw [1] e o propagador da teoria MCFJ foi obtido por [2]. É necessário deixar bem claro que a intenção deste capítulo não é esmiuçar a teoria de MCFJ, mas apenas familiarizar o leitor com o método em um cenário mais elementar que o gravitacional. Dessa forma, não iremos analisar a estrutura dos polos obtidos a partir do propagador encontrado, daremos enfoque apenas ao formalismo de projetores tensoriais empregado no cálculo do propagador, bem como a extensão da base de projetores, algo que será recorrente nos capítulos que se seguem.

1.1 Propagador para a teoria de Maxwell-Carrol-Field-Jackiw

A ação do modelo de MFCJ é dada por

$$\mathcal{L} = -\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} - \frac{1}{4}\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta F_{\rho\varphi} + \frac{1}{2\xi}(\partial_\mu A^\mu)^2, \tag{1.1.1}$$

onde A^μ é o 4-potencial, $V^\mu = (V_0, \mathbf{V})$ é o 4-vetor fixo de violação da simetria de Lorentz, $\varepsilon^{\mu\nu\alpha\beta}$ é o símbolo de Levi-Civita, $F_{\mu\nu}$ é o tensor do campo de Maxwell e a última parcela de (1.1.1) é o chamado "gauge-fixing". Aqui o nosso objetivo é reescrever a eq.(1.1.1) em uma forma conhecida como "quadrática", o que significa escrevê-la na forma

$$\mathcal{L} = A^\mu \mathcal{D}_{\mu\nu} A^\nu, \tag{1.1.2}$$

onde $\mathcal{D}_{\mu\nu}$ é um operador tensorial. Podemos fazer isso reescrevendo, cada um dos termos que compõem a lagrangiana (1.1.1):

$$F_{\alpha\beta}F^{\alpha\beta} = (\partial_\alpha A_\beta - \partial_\beta A_\alpha)\left(\partial^\alpha A^\beta - \partial^\beta A^\alpha\right) = 2\partial_\alpha A_\beta \partial^\alpha A^\beta - 2\partial_\alpha A_\beta \partial^\beta A^\alpha. \tag{1.1.3}$$

Vamos usar a derivada total para reescrever cada uma destas parcelas,

$$\partial_\alpha\left(A_\beta\partial^\alpha A^\beta\right)=\partial_\alpha A_\beta\partial^\alpha A^\beta+A_\beta\partial_\alpha\partial^\alpha A^\beta, \tag{1.1.4}$$
$$\partial_\alpha\left(A_\beta\partial^\beta A^\alpha\right)=\partial_\alpha A_\beta\partial^\beta A^\alpha+A_\beta\partial_\alpha\partial^\beta A^\alpha, \tag{1.1.5}$$

como a derivada total não contribui para as equações de movimento, temos

$$F_{\alpha\beta}F^{\alpha\beta}=-2A^\alpha\eta_{\alpha\beta}\Box A^\beta+2A^\beta\partial_\alpha\partial_\beta A^\alpha=-2A^\alpha\theta_{\alpha\beta}A^\beta, \tag{1.1.6}$$

onde

$$\theta_{\alpha\beta}=\eta_{\alpha\beta}-\omega_{\alpha\beta},\quad \omega_{\alpha\beta}=\frac{\partial_\alpha\partial_\beta}{\Box}, \tag{1.1.7}$$

são os projetores transversal e longitudinal respectivamente. Finalmente,

$$-\frac{1}{4}F_{\alpha\beta}F^{\alpha\beta}=\frac{1}{2}A^\beta\Box\theta_{\alpha\beta}A^\alpha. \tag{1.1.8}$$

Para a próxima parcela, temos

$$\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta F_{\rho\varphi}=\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta\left(\partial_\rho A_\varphi-\partial_\varphi A_\rho\right), \tag{1.1.9}$$
$$=\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta\partial_\rho A_\varphi-\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta\partial_\varphi A_\rho. \tag{1.1.10}$$

Usando as simetrias do Símbolo de Levi-Civita, podemos obter

$$\varepsilon^{\alpha\beta\varphi\rho}V_\alpha A_\beta\partial_\varphi A_\rho=-\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta\partial_\varphi A_\rho-\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta\partial_\varphi A_\rho, \tag{1.1.11}$$
$$=-2A_\beta\varepsilon^{\alpha\beta\rho\varphi}V_\alpha\partial_\varphi A_\rho=-2A_\beta S^{\beta\rho}A_\rho, \tag{1.1.12}$$

sendo

$$S^{\beta\rho}=\varepsilon^{\alpha\beta\rho\varphi}V_\alpha\partial_\varphi \tag{1.1.13}$$

Vamos mudar a configuração dos índices por questão de uniformização com o padrão inicialmente escrito na eq. (1.1.8)

$$-\frac{1}{4}\varepsilon^{\alpha\beta\rho\varphi}V_\alpha A_\beta F_{\rho\varphi}=\frac{1}{2}A^\alpha S_{\alpha\beta}A^\beta. \tag{1.1.14}$$

Por último, tratamos o termo de "gauge-fixing",

$$\frac{1}{2\xi}(\partial_\mu A^\mu)^2=\frac{1}{2\xi}(A^\beta\partial_\beta\partial_\alpha A^\alpha)=-\frac{1}{2\xi}(A^\beta\Box\omega_{\alpha\beta}A^\alpha). \tag{1.1.15}$$

A eq.(1.1.13) nos traz um operador tipo Chern-Simons em $(3+1)$ dimensões. Ressaltamos que além de evidenciar a presença do campo de fundo V_φ, tal operador é antissimétrico. Substituindo as eqs.(1.1.8), (1.1.14) e (1.1.15) na eq.(1.1.1), resulta:

$$\mathcal{L}=\frac{1}{2}A^\alpha\mathcal{D}_{\alpha\beta}A^\beta, \tag{1.1.16}$$

onde o operador $\mathcal{D}_{\alpha\beta}$, dado por:

$$\mathcal{D}_{\alpha\beta}=\Box\theta_{\alpha\beta}-\frac{\Box}{\xi}\omega_{\alpha\beta}+S_{\alpha\beta}, \tag{1.1.17}$$

possui uma representação matricial própria. A sua forma inversa $\mathcal{D}^{-1}_{\alpha\beta}$ é o núcleo do propagador que desejamos encontrar:

$$\mathcal{D}^{\alpha\beta}\left(\mathcal{D}_{\alpha\nu}\right)^{-1} = \delta_{\nu}{}^{\beta}, \tag{1.1.18}$$

$$\mathcal{D}^{\alpha}{}_{\beta}\left(\mathcal{D}_{\alpha\nu}\right)^{-1} = \eta_{\nu\beta}. \tag{1.1.19}$$

Analisando a eq.(1.1.17) podemos inferir, a princípio, uma forma de escrever a inversa de $\mathcal{D}_{\alpha\beta}$. Esta seria composta por uma combinação linear dos operadores $[\omega_{\alpha\beta},\, \theta_{\alpha\beta},\, S_{\alpha\beta}]$, ou seja,

$$\left(\mathcal{D}_{\alpha\nu}\right)^{-1} = a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu}. \tag{1.1.20}$$

Assim, o operador $\left(\mathcal{D}_{\alpha\beta}\right)^{-1}$ pode ser obtido ao se determinar os coeficientes a_1, b_1 e c_1. Para encontrar estes coeficientes, substituimos (1.1.20) na Eq. (1.1.19):

$$\mathcal{D}^{\alpha}{}_{\beta}\mathcal{D}^{-1}{}_{\alpha\nu} = \left(\Box\theta^{\alpha}{}_{\beta} - \frac{\Box}{\xi}\omega^{\alpha}{}_{\beta} + S^{\alpha}{}_{\beta}\right)\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu}\right),$$

$$\begin{aligned}\mathcal{D}^{\alpha}{}_{\beta}\mathcal{D}^{-1}{}_{\alpha\nu} = \Box\theta^{\alpha}{}_{\beta}\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu}\right) - \frac{\Box}{\xi}\omega^{\alpha}{}_{\beta}\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu}\right)\\ + S^{\alpha}{}_{\beta}\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu}\right).\end{aligned} \tag{1.1.21}$$

As contrações tensoriais existentes na expressão acima podem ser determinadas conhecendo a álgebra dos operadores, que apresentamos a seguir:

$$\omega_{\alpha\beta}\omega^{\alpha}{}_{\nu} = \frac{\partial_\alpha\partial_\beta}{\Box}\frac{\partial^\alpha\partial_\nu}{\Box} = \omega_{\nu\beta}. \tag{1.1.22}$$

$$\theta_{\alpha\beta}\omega^{\alpha}{}_{\nu} = \omega_{\alpha\beta}\theta^{\alpha}{}_{\nu} = \eta_{\alpha\beta}\omega^{\alpha}{}_{\nu} - \omega_{\alpha\beta}\omega^{\alpha}{}_{\nu} = \omega_{\beta\nu} - \omega_{\beta\nu} = 0, \tag{1.1.23}$$

$$\omega_{\alpha\beta}S^{\alpha}{}_{\nu} = S_{\alpha\beta}\omega^{\alpha}{}_{\nu} = \frac{\partial_\alpha\partial_\beta}{\Box}\eta_{\nu\kappa}\varepsilon^{\alpha\kappa\rho\varphi}V_\rho\partial_\varphi = 0, \tag{1.1.24}$$

por causa da antissimetria do símbolo de Levi-Civita nos índices α e φ. Seguimos:

$$\theta_{\alpha\beta}\theta^{\alpha}{}_{\nu} = \eta_{\alpha\beta}\theta^{\alpha}{}_{\nu} - \omega_{\alpha\beta}\theta^{\alpha}{}_{\nu} = \theta_{\nu\beta}. \tag{1.1.25}$$

$$\theta_{\alpha\beta}S^{\alpha}{}_{\nu} = S_{\alpha\beta}\theta^{\alpha}{}_{\nu} = S_{\alpha\beta}\left(\delta^{\alpha}{}_{\nu} - \omega^{\alpha}{}_{\nu}\right) = \varepsilon_{\alpha\beta\rho\varphi}V^\rho\partial^\varphi\delta^{\alpha}{}_{\nu} = S_{\nu\beta}. \tag{1.1.26}$$

$$S_{\alpha\beta}S^{\alpha}{}_{\nu} = \eta_{\kappa\nu}S_{\alpha\beta}S^{\alpha\kappa} = \eta_{\kappa\nu}\varepsilon_{\alpha\beta\rho\varphi}V^\rho\partial^\varphi\varepsilon^{\alpha\kappa\sigma\lambda}V_\sigma\partial_\lambda. \tag{1.1.27}$$

Para finalizar, devemos conhecer o resultado

$$\varepsilon^{\alpha\kappa\sigma\lambda}\varepsilon_{\alpha\beta\rho\varphi} = \delta^{\kappa}{}_{\beta}\left(\delta^{\sigma}{}_{\rho}\delta^{\lambda}{}_{\varphi} - \delta^{\sigma}{}_{\varphi}\delta^{\lambda}{}_{\rho}\right) + \delta^{\kappa}{}_{\rho}\left(\delta^{\sigma}{}_{\varphi}\delta^{\lambda}{}_{\beta} - \delta^{\sigma}{}_{\beta}\delta^{\lambda}{}_{\varphi}\right) + \delta^{\kappa}{}_{\varphi}\left(\delta^{\sigma}{}_{\beta}\delta^{\lambda}{}_{\rho} - \delta^{\sigma}{}_{\rho}\delta^{\lambda}{}_{\beta}\right), \tag{1.1.28}$$

que leva ao desenvolvimento:

$$\begin{aligned}\varepsilon^{\alpha\kappa\sigma\lambda}\varepsilon_{\alpha\beta\rho\varphi}V_\sigma\partial_\lambda V^\rho\partial^\varphi = \delta^{\kappa}{}_{\beta}\left(V_\rho\partial_\varphi V^\rho\partial^\varphi - V_\sigma\partial_\lambda V^\lambda\partial^\sigma\right)\\ + \left(V_\varphi\partial_\beta V^\kappa\partial^\varphi - V_\beta\partial_\lambda V^\kappa\partial^\lambda\right) + \left(V_\beta\partial_\rho V^\rho\partial^\kappa - V_\rho\partial_\beta V^\rho\partial^\kappa\right),\end{aligned} \tag{1.1.29}$$

$$S_{\alpha\beta}S^{\alpha}{}_{\nu} = \eta_{\beta\nu}\left[V^2\Box - \left(V\cdot\partial\right)^2\right] + \left[\left(V\cdot\partial\right)\partial_\beta V_\nu - V_\beta V_\nu\Box\right] + \left(V_\beta V^\rho\partial_\rho\partial_\nu - V^2\partial_\beta\partial_\nu\right), \tag{1.1.30}$$

$$= \left(\theta_{\nu\beta} + \omega_{\nu\beta}\right)\left(V^2\Box - \lambda^2\right) - \Box\Lambda_{\nu\beta} + \lambda\left(\Sigma_{\nu\beta} + \Sigma_{\beta\nu}\right) - V^2\Box\omega_{\nu\beta}, \tag{1.1.31}$$

$$S_{\alpha\beta}S^{\alpha}{}_{\nu} = \theta_{\nu\beta}\left(V^2\Box - \lambda^2\right) - \lambda^2\omega_{\nu\beta} + \lambda\tilde{\Sigma}_{\nu\beta} - \Box\Lambda_{\nu\beta} = f_{\nu\beta}, \tag{1.1.32}$$

onde definimos

$$\Sigma_{\nu\beta} = V_\nu \partial_\beta,\ V_\sigma \partial^\sigma = \lambda,\ \tilde{\Sigma}_{\nu\beta} = \Sigma_{\nu\beta} + \Sigma_{\beta\nu},\ \Lambda_{\nu\beta} = V_\nu V_\beta. \tag{1.1.33}$$

O surgimento de termos novos envolvendo as combinações $V_\mu \partial_\nu$ e $V_\mu V_\nu$, indica que necessitamos introduzir os elementos $\left[\tilde{\Sigma}_{\mu\nu} \text{ e } \Lambda_{\mu\nu}\right]$ no conjuntode projetores inicial: $[\omega_{\alpha\beta},\ \theta_{\alpha\beta},\ S_{\alpha\beta}]$, uma vez que a álgebra dos operadores inicialmente proposta não é fechada. Logo, o operador inverso deve ser proposto numa forma mais ampla, a saber:

$$\mathcal{D}^{-1}{}_{\alpha\nu} = a_1 \omega_{\alpha\nu} + b_1 \theta_{\alpha\nu} + c_1 S_{\alpha\nu} + d_1 \tilde{\Sigma}_{\alpha\nu} + e_1 \Lambda_{\alpha\nu}. \tag{1.1.34}$$

Vamos calcular as novas contrações iniciando com $\tilde{\Sigma}_{\mu\nu}$

$$\tilde{\Sigma}_{\alpha\beta} \omega^\alpha{}_\nu = \left(V_\alpha \partial_\beta + V_\beta \partial_\alpha\right) \omega^\alpha{}_\nu = \lambda \omega_{\nu\beta} + \Sigma_{\beta\nu}, \tag{1.1.35}$$

$$\tilde{\Sigma}_{\alpha\beta} \theta^\alpha{}_\nu = \tilde{\Sigma}_{\alpha\beta} \left(\delta^\alpha{}_\nu - \omega^\alpha{}_\nu\right) = \tilde{\Sigma}_{\nu\beta} - \lambda \omega_{\nu\beta} - \Sigma_{\beta\nu}, \tag{1.1.36}$$

$$\tilde{\Sigma}_{\alpha\beta} S^\alpha{}_\nu = \tilde{\Sigma}_{\alpha\beta} \eta_{\nu\kappa} S^{\alpha\nu} = \eta_{\nu\kappa} \left(V_\alpha \partial_\beta + V_\beta \partial_\alpha\right) \varepsilon^{\alpha\nu\kappa\lambda} V_\kappa \partial_\lambda = 0, \tag{1.1.37}$$

$$\Lambda_{\alpha\beta} \omega^\alpha{}_\nu = \frac{\lambda}{\Box} \Sigma_{\beta\nu}, \tag{1.1.38}$$

$$\Lambda_{\alpha\beta} \theta^\alpha{}_\nu = \Lambda_{\alpha\beta} \left(\delta^\alpha{}_\nu - \omega^\alpha{}_\nu\right) = \Lambda_{\nu\beta} - \frac{\lambda}{\Box} \Sigma_{\beta\nu}, \tag{1.1.39}$$

$$\Lambda_{\alpha\beta} S^\alpha{}_\nu = \Lambda_{\alpha\beta} \eta_{\nu\kappa} S^{\alpha\nu} = V_\alpha V_\beta \varepsilon^{\alpha\nu\kappa\lambda} V_\kappa \partial_\lambda = 0, \tag{1.1.40}$$

$$\tilde{\Sigma}_{\alpha\beta} \Lambda^\alpha{}_\nu = \left(V_\alpha \partial_\beta + V_\beta \partial_\alpha\right) V^\alpha V_\nu = V^2 \Sigma_{\nu\beta} + \lambda \Lambda_{\nu\beta}, \tag{1.1.41}$$

$$\tilde{\Sigma}_{\alpha\beta} \tilde{\Sigma}^\alpha{}_\nu = \left(V_\alpha \partial_\beta + V_\beta \partial_\alpha\right) \left(V^\alpha \partial_\nu + V_\nu \partial^\alpha\right) = \Box V^2 \omega_{\nu\beta} + \lambda \tilde{\Sigma}_{\nu\beta} + \Box \Lambda_{\nu\beta}, \tag{1.1.42}$$

$$\Lambda_{\alpha\beta} \Lambda^\alpha{}_\nu = V^2 \Lambda_{\nu\beta}. \tag{1.1.43}$$

Podemos organizar esses resultados na tabela a seguir.

	$\omega^\alpha{}_\nu$	$\theta^\alpha{}_\nu$	$S^\alpha{}_\nu$	$\tilde{\Sigma}^\alpha{}_\nu$	$\Lambda^\alpha{}_\nu$
$\omega_{\alpha\beta}$	$\omega_{\nu\beta}$	0	0	$\lambda\omega_{\nu\beta} + \Sigma_{\nu\beta}$	$\frac{\lambda}{\Box}\Sigma_{\nu\beta}$
$\theta_{\alpha\beta}$	0	$\theta_{\nu\beta}$	$S_{\nu\beta}$	$\tilde{\Sigma}_{\nu\beta} - \lambda\omega_{\nu\beta} - \Sigma_{\nu\beta}$	$\Lambda_{\nu\beta} - \frac{\lambda}{\Box}\Sigma_{\nu\beta}$
$S_{\alpha\beta}$	0	$S_{\nu\beta}$	$f_{\nu\beta}$	0	0
$\tilde{\Sigma}_{\alpha\beta}$	$\lambda\omega_{\nu\beta} + \Sigma_{\beta\nu}$	$\tilde{\Sigma}_{\nu\beta} - \lambda\omega_{\nu\beta} - \Sigma_{\beta\nu}$	0	$\Box V^2\omega_{\nu\beta} + \lambda\tilde{\Sigma}_{\nu\beta} + \Box\Lambda_{\nu\beta}$	$V^2\Sigma_{\nu\beta} + \lambda\Lambda_{\nu\beta}$
$\Lambda_{\alpha\beta}$	$\frac{\lambda}{\Box}\Sigma_{\beta\nu}$	$\Lambda_{\nu\beta} - \frac{\lambda}{\Box}\Sigma_{\beta\nu}$	0	$V^2\Sigma_{\beta\nu} + \lambda\Lambda_{\nu\beta}$	$V^2\Lambda_{\nu\beta}$

Note que todas as contrações calculadas enquadram-se no conjunto já proposto: $[\omega_{\alpha\beta},\ \theta_{\alpha\beta},\ S_{\alpha\beta},\ \tilde{\Sigma}_{\alpha\beta}$ e $\Lambda_{\alpha\beta}]$. Se no processo de obtenção das contrações deste conjunto surgisse uma outra estrutura diferente das inicialmente supostas, seria necessário introduzir este novo elemento no conjunto e repetir a tarefa de calcular as contrações (até que nada novo surgisse).

A álgebra fechada obtida em [2] contém um total de 6 operadores, conseguimos fechá-la com apenas 5, o que se deve ao fato de considerarmos o projetor $\tilde{\Sigma}_{\alpha\beta}$, que congrega dois projetores de [2]: $\Sigma_{\alpha\beta}$ e $\Sigma_{\beta\alpha}$. A vantagem é que o projetor $\tilde{\Sigma}_{\alpha\beta}$ é simétrico, enquanto $\Sigma_{\alpha\beta}$ e $\Sigma_{\beta\alpha}$ não possuem simetria definida. Assim, obtemos

$$\mathcal{D}^\alpha{}_\beta \mathcal{D}^{-1}{}_{\alpha\nu} = \eta_{\nu\beta} = \left(\Box\theta^\alpha{}_\beta - \frac{\Box}{\xi}\omega^\alpha{}_\beta + S^\alpha{}_\beta\right)\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu} + d_1\tilde{\Sigma}_{\alpha\nu} + e_1\Lambda_{\alpha\nu}\right), \tag{1.1.44}$$

$$\begin{aligned}\mathcal{D}^{\alpha}{}_{\beta}\mathcal{D}^{-1}{}_{\alpha\nu} = \Box\theta^{\alpha}{}_{\beta}&\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu} + d_1\tilde{\Sigma}_{\alpha\nu} + e_1\Lambda_{\alpha\nu}\right)\\ &- \frac{\Box}{\xi}\omega^{\alpha}{}_{\beta}\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu} + d_1\tilde{\Sigma}_{\alpha\nu} + e_1\Lambda_{\alpha\nu}\right)\\ &+ S^{\alpha}{}_{\beta}\left(a_1\omega_{\alpha\nu} + b_1\theta_{\alpha\nu} + c_1 S_{\alpha\nu} + d_1\tilde{\Sigma}_{\alpha\nu} + e_1\Lambda_{\alpha\nu}\right), \end{aligned}\tag{1.1.45}$$

$$\begin{aligned}\mathcal{D}^{\alpha}{}_{\beta}\mathcal{D}^{-1}{}_{\alpha\nu} = b_1\Box\theta_{\nu\beta} &+ c_1\Box S_{\nu\beta} + d_1\Box\left(\tilde{\Sigma}_{\nu\beta} - \lambda\omega_{\nu\beta} - \Sigma_{\nu\beta}\right) + e_1\left(\Box\Lambda_{\nu\beta} - \lambda\Sigma_{\nu\beta}\right)\\ &- \frac{1}{\xi}\left[a_1\Box\omega_{\nu\beta} + d_1\Box\left(\lambda\omega_{\nu\beta} + \Sigma_{\nu\beta}\right) + e_1\lambda\Sigma_{\nu\beta}\right]\\ &+ b_1 S_{\nu\beta} + c_1\left[\theta_{\nu\beta}\left(V^2\Box - \lambda^2\right) - \lambda^2\omega_{\nu\beta} + \lambda\tilde{\Sigma}_{\nu\beta} - \Box\Lambda_{\nu\beta}\right].\end{aligned}\tag{1.1.46}$$

Agrupando os termos semelhantes,

$$\begin{aligned}\mathcal{D}^{\alpha}{}_{\beta}\mathcal{D}^{-1}{}_{\alpha\nu} = &\left[b_1\Box + c_1\left(V^2\Box - \lambda^2\right)\right]\theta_{\nu\beta}\\ &+ \left[-d_1\Box\lambda - \frac{1}{\xi}a_1\Box - c_1\lambda^2 - \frac{1}{\xi}d_1\Box\lambda\right]\omega_{\nu\beta} + \left[b_1 + c_1\Box\right]S_{\nu\beta}\\ &+ \left[c_1\lambda - \frac{1}{\xi}d_1\Box - \frac{1}{\xi}e_1\lambda - e_1\lambda\right]\Sigma_{\nu\beta} + \left[d_1\Box + c_1\lambda\right]\Sigma_{\beta\nu} + \left[e_1\Box - c_1\Box\right]\Lambda_{\nu\beta}.\end{aligned}\tag{1.1.47}$$

Substituindo o resultado (1.1.47) na eq. (1.1.19) e, usando a relação,

$$\eta_{\nu\beta} = \theta_{\nu\beta} + \omega_{\nu\beta},\tag{1.1.48}$$

podemos coletar os termos semelhantes nos projetores $\theta_{\nu\beta}$, $\omega_{\nu\beta}$, $S_{\nu\beta}$, $\Sigma_{\nu\beta}$, $\Sigma_{\beta\nu}$ e $\Lambda_{\nu\beta}$, comparando-os com a eq. (1.1.48), de modo que escrevemos o sistema

$$b_1\Box + c_1\left(V^2\Box - \lambda^2\right) = 1,\tag{1.1.49}$$

$$d_1\Box\lambda + \frac{1}{\xi}a_1\Box + c_1\lambda^2 + \frac{1}{\xi}d_1\Box\lambda = -1,\tag{1.1.50}$$

$$b_1 + c_1\Box = 0,\tag{1.1.51}$$

$$c_1\lambda - \frac{1}{\xi}d_1\Box - \frac{1}{\xi}e_1\lambda - e_1\lambda = 0,\tag{1.1.52}$$

$$d_1\Box + c_1\lambda = 0,\tag{1.1.53}$$

$$e_1\Box - c_1\Box = 0.\tag{1.1.54}$$

A solução deste sistema é

$$a_1 = -\frac{\xi}{\Box} - \frac{\lambda^2}{\Box\left(\Box^2 - \Box V^2 + \lambda^2\right)}, \qquad b_1 = \frac{\Box}{\Box^2 - \Box V^2 + \lambda^2},\tag{1.1.55}$$

$$c_1 = -\frac{1}{\Box^2 - \Box V^2 + \lambda^2}, \qquad d_1 = \frac{\lambda}{\Box\left(\Box^2 - \Box V^2 + \lambda^2\right)},\tag{1.1.56}$$

$$e_1 = -\frac{1}{\Box^2 - \Box V^2 + \lambda^2}.\tag{1.1.57}$$

De posse desses resultados, o operador $\left(\mathcal{D}_{\alpha\beta}\right)^{-1}$ assume a forma:

$$\begin{aligned}\left(\mathcal{D}_{\alpha\beta}\right)^{-1} = &-\left[\frac{\xi}{\Box} + \frac{\lambda^2}{\Box\left(\Box^2 - \Box V^2 + \lambda^2\right)}\right]\omega_{\alpha\beta} + \left[\frac{\Box}{\Box^2 - \Box V^2 + \lambda^2}\right]\theta_{\alpha\nu}\\ &- \frac{S_{\alpha\beta}}{\Box^2 - \Box V^2 + \lambda^2} + \frac{\lambda}{\Box\left(\Box^2 - \Box V^2 + \lambda^2\right)}\tilde{\Sigma}_{\alpha\beta} - \frac{\Lambda_{\alpha\beta}}{\Box^2 - \Box V^2 + \lambda^2},\end{aligned}\tag{1.1.58}$$

ou de forma simplificada

$$(\mathcal{D}_{\alpha\beta})^{-1}=\frac{1}{\Box\left(\Box^{2}-\Box V^{2}+\lambda^{2}\right)}\left\{\Box^{2}\theta_{\alpha\nu}-\left[\xi\left(\Box^{2}-\Box V^{2}+\lambda^{2}\right)+\lambda^{2}\right]\omega_{\alpha\beta}-\Box\left(S_{\alpha\beta}+\Lambda_{\alpha\beta}\right)+\lambda\tilde{\Sigma}_{\alpha\beta}\right\}. \tag{1.1.59}$$

Escrevemos então o propagador de Feynman da teoria de Maxwell-Carrol-Field-Jackiw no espaço dos momentos $\triangle_{\alpha\beta}=i\left(\mathcal{D}_{\alpha\beta}\right)^{-1}$:

$$\begin{aligned}\triangle_{\alpha\beta}=-\frac{i}{p^{2}\left(p^{4}+p^{2}V^{2}-(V\cdot p)^{2}\right)}&\left\{p^{4}\theta_{\alpha\nu}-\left[\xi\left(p^{4}+p^{2}V^{2}-(V\cdot p)^{2}\right)+\lambda^{2}\right]\omega_{\alpha\beta}\right.\\&\left.+\,p^{2}\left[S_{\alpha\beta}\left(p\right)+\Lambda_{\alpha\beta}\right]-\left(V\cdot p\right)\tilde{\Sigma}_{\alpha\beta}\left(p\right)\right\},\end{aligned} \tag{1.1.60}$$

onde

$$S^{\alpha\beta}\left(p\right)=-i\varepsilon^{\alpha\beta\varphi\rho}V_{\varphi}p_{\rho},\qquad\tilde{\Sigma}_{\alpha\beta}\left(p\right)=V_{\alpha}p_{\beta}+V_{\beta}p_{\alpha}. \tag{1.1.61}$$

Os polos do propagador são

$$p^{2}=0, \tag{1.1.62}$$

$$p^{4}+p^{2}V^{2}-(V\cdot p)^{2}=0. \tag{1.1.63}$$

CAPÍTULO 2

Propagador para teorias gravitacionais

Há um grande interesse físico no propagador de uma teoria, em razão das informações que podem ser extraídas do mesmo, a exemplo das relações de dispersão que descrevem como a energia de um campo depende do momento linear. A causalidade também é um outro aspecto que pode ser obtido a partir das relações de dispersão presentes no propagador. Na verdade, a causalidade pode ser investigada e examinada a partir das relações de dispersão extraídas do propagador, sendo uma propriedade importante para estabelecer a consistência de uma teoria de campos no que tange à propagação de sinais (transmissão de informação), em atendimento ao *Princípio da Causalidade*.

O cálculo do propagador requer a utilização de uma base de projetores. Para teorias eletromagnéticas, de spin-1, os chamados projetores transversal ($\theta_{\mu\nu}$) e longitudinal ($\omega_{\mu\nu}$) compõem essa base, cuja denominação se origina da forma como atuam no momento p^{μ}, como veremos na tabela 2.1. Apesar de ser útil para o campo eletromagnético, esta base de projetores não é adequada para o campo gravitacional, que possui spin-2. No caso eletromagnético, o spin do campo em questão (fóton) é 1, e a forma bilinear da lagrangiana será compatível com projetores de dois índices; no cenário gravitacional, a lagrangiana do campo de spin-2 conduz a uma forma bilinear com 4 índices tensoriais. A base de projetores adequada neste caso é a de Barnes-Rivers [3] e [4]: composta por um conjunto de combinações lineares dos projetores transversal e longitudinal linearmente independentes entre si, todos constituídos por estruturas de quatro índices tensoriais.

2.1 A teoria do campo gravitacional

A teoria da relatividade geral de Einstein (TRG) [5] possui um dos legados mais bem sucedidos na história da Física. Enquanto a teoria da relatividade restrita (TRR) surgiu na tentativa de compatibilizar resultados experimentais de Michelson-Morley [6], a TRG teve origem em tentativas de conciliar a TRR com a teoria de gravitação de Newton. Tentar conciliar essas duas teorias esbarrava em um problema: associar a estrutura geométrica do espaço-tempo com a gravidade [7]. A TRR é uma teoria baseada no conceito de espaço-tempo de Minkowski: uma estrutura de quatro $(3 + 1)$ dimensões (três espaciais e um temporal), na qual espaço e tempo passam a ser interpendentes. A quarta dimensão, o tempo, possui natureza distinta das demais e isso se traduz no sinal negativo que surge quando escrevemos o elemento de distância no espaço de Minkowski [8],

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -c^2 dt^2 + dx^2 + dy^2 + dz^2, \tag{2.1.1}$$

onde $\eta_{\mu\nu}$ é a métrica de Minkowski com assinatura $(-, +, +, +)$, dx^μ é um elemento infinitesimal de de variação do vetor posição e c a velocidade da luz no vácuo.

Podemos notar que a contrução da TRR foi realizada por meio de dois postulados fundamentais e possui consequências geométricas como a relatividade da simultaneidade, dilatação temporal e contração espacial, enquanto que a gravitação newtoniana foi formulada em um cenário de forças de atração. O campo $\vec{g}$ seria a origem dessa força

$$\vec{g} = \frac{GM}{r^2}\hat{r}, \tag{2.1.2}$$

gerado por um objeto com massa M, G a constante de gravitação universal e r a distância entre os corpos que interagem. Este campo tem origem no seguinte potencial

$$\phi(r) = -\frac{GM}{r}, \tag{2.1.3}$$

que é a quantidade básica na gravitação de Newton. Aqui é importante mencionar que os cenários sob a ação da gravitação não podem ser descritos pela TRR, uma vez que os corpos sujeitos à interação gravitacional estão acelerados e não se enquadram como referenciais inerciais ademais, a gravitação newtoniana não explica a origem do campo gravitacional. Ao apenas descrever como os corpos interagem e não explicar a origem da interação, a gravitação de Newton deixa a desejar em sua base conceitual. Nesse sentido, a princípio, uma conexão entre a TRR e a gravitação deve conter duas principais características: ressignificação do conceito de campo gravitacional, explicando a origem do campo gravitacional e o estabelecimento de um novo princípio de relatividade, que abrangesse os referenciais não inerciais.

2.1.1 Princípio de equivalência

Na TRG, a geometria do espaço-tempo define a dinâmica do campo que rege e transmite a interação gravitacional. Esta geometria é representada pelo tensor métrico ($g^{\mu\nu}$), que é quem assume a função de descrever a gravidade como campo propagante de caráter universal, ou seja, tudo que existe interage por meio da gravidade.

O estabelecimento de sistemas de referência é a primeira tarefa a ser realizada quando se deseja medir alguma grandeza física. A própria TRR surge da necessidade de relacionar as medidas feitas por observadores localizados em sistemas distintos sem discrepância, pois diferentes observadores devem obter os mesmos resultados para experimentos do mesmo tipo. O primeiro postulado da relatividade de Einstein sintetiza o caráter imutável das leis físicas frente à mudança de referencial inercial.

De modo análogo, o *princípio de equivalência* advém de uma profunda reflexão acerca da igualdade entre massa inercial e massa gravitacional [5], este princípio afirma que experimentos em um laboratório em queda livre fornecem resultados que são indistinguíveis dos mesmos experimentos realizados em um referencial inercial no espaço vazio. A massa inercial ter o mesmo valor que a massa gravitacional era uma ideia já considerada por Newton, isso reforça ainda mais a base conceitual para o estabelecimento do princípio de equivalência que surge da problemática na definição de referencial inercial em um cenário de interação que tudo afeta. A origem desse pensamento começou por volta de 1907 quando o Einstein questionou a si mesmo a respeito de como a gravitação de Newton deveria ser ajustada para ser compatível com a relatividade restrita. Neste momento teve o que ele próprio chamou de “pensamento mais feliz da minha vida” ao imaginar que uma pessoa em queda livre não sentiria o próprio peso. Portanto, a igualdade entre as massas inercial e gravitacional não pode ser mera coincidência e que esta deveria levar a uma ampla compreensão da inércia e da gravitação.

Dentro da gravitação newtoniana, ter massa é um requisito para que um objeto perceba fenômenos gravitacionais, a luz, assim, não se enquadraria. No entanto a natureza se manifesta de forma distinta ao ideal newtoniano e a luz interage com campos gravitacionais. Essa interação não ocorre porque a luz possua massa, mas o espaço-tempo que foi deformado pela presença de um corpo massivo. Imagine uma folha de caderno com pautas sobre uma mesa, as linhas presentes nessa folha representam a trajetória de um feixe de luz se propagando numa região do espaço-tempo suficientemente distante de qualquer objeto capaz de produzir campo gravitacional. Agora, se apenas segurarmos a folha pelos lados, sem esticar, e a levantarmos, vamos observar a formação de uma pequena depressão no meio da folha, e as linhas que antes eram retas foram curvadas pelo peso da folha de papel. As linhas ainda representam o caminho do feixe de luz, mas este deixou de seguir em linha reta e sofreu um desvio. Este é um dos pontos mais delicados na TRG, pois ao considerarmos que um feixe de luz sofre um desvio na sua trajetória quando observada por um referencial acelerado, tendo em vista o princípio de equivalência, encontramos uma abertura para a verificação experimental da TRG. Isso quer dizer que se a luz é desviada por objetos acelerados, será desviada por campos gravitacionais e isso poderia ser testado, basta haver um objeto com massa suficiente para gerar um campo gravitacional capaz de alterar a trajetória da luz e um dispositivo sensível o suficiente para observar esta alteração. Tal experiência foi realizada em 1919, na cidade de Sobral-CE. Uma expedição confirmou com certo grau de precisão, a validade do princípio de equivalência sendo o começo do legado de uma das teorias que melhor performaram na explicação de fenômenos conhecidos, como, por exemplo, a precessão do periélio de Mercúrio e previsão de coisas que até então seriam inimagináveis como as ondas gravitacionais e buracos negros.

2.1.2 Gravidade geométrica

A TRG descreve a gravitação como um efeito da curvatura do espaço-tempo [9]. De acordo com o princípio da equivalência, a gravidade pode ser anulada localmente, pois é possível definir um observador em queda livre, que é localmente equivalente a um observador inercial. É como se tal observador experimentasse uma geometria de curvatura nula (localmente). Uma variedade é justamente a entidade matemática que satisfaz essas condições, ou seja, a variedade é um espaço continuamente curvo que localmente parece euclidiano. Por exemplo, em curtas escalas de distância, podemos afirmar que a superfície terrestre é plana o que não é verdade de modo global.

As equações de campo de Einstein são fundamentais no estudo da TRG, estando para a relatividade de Einstein como as equações de Maxwell estão para o eletromagnetismo. Podem ser obtidas por meio do princípio de minimização aplicado sobre a ação de Einstein-Hilbert,

$$\mathcal{S} = \int \left(\frac{R}{2\kappa} + \mathcal{L}_M \right) \sqrt{-g} d^4x, \tag{2.1.4}$$

onde R é o escalar de Ricci, $\kappa = 8\pi G c^{-4}$ e $\mathcal{L}_M$ representa a densidade lagrangiana que contém a informação sobre a distribuição de matéria e energia. A variação da ação é inicialmente escreita como:

$$\delta\mathcal{S} = \frac{1}{2\kappa} \int \delta \left(\sqrt{-g} R \right) d^4x + \int \delta \left(\sqrt{-g} \mathcal{L}_M \right) d^4x. \tag{2.1.5}$$

Vamos calcular termo a termo. Primeiramente,

$$\delta\left(\sqrt{-g}R\right)=\sqrt{-g}\delta R+R\delta\left(\sqrt{-g}\right), \tag{2.1.6}$$

que foi obtido no apêndice A. Assim, podemos reescrever a primeira integral em (2.1.5) como:

$$\begin{aligned}\frac{1}{2\kappa}\int\delta\left(\sqrt{-g}R\right)d^4x=\frac{1}{2\kappa}\int\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}d^4x\\+\frac{1}{2\kappa}\int\nabla_\alpha\left(g^{\mu\nu}\delta\Gamma^{\alpha}{}_{\mu\nu}-g^{\mu\alpha}\delta\Gamma^{\lambda}{}_{\mu\lambda}\right)\sqrt{-g}d^4x,\end{aligned}$$

onde fizemos uma mudança nos índices contraídos. A segunda dessa integrais é nula, por se tratar de uma derivada total, nos restando apenas calcular a variação para o termo $\mathcal{L}_M\sqrt{-g}$

$$\delta\left(\mathcal{L}_M\sqrt{-g}\right)=\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}+\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\delta\left(\partial_\alpha g^{\mu\nu}\right) \tag{2.1.7}$$

$$=\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}+\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\partial_\alpha\left(\delta g^{\mu\nu}\right) \tag{2.1.8}$$

note que

$$\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\delta g^{\mu\nu}\right]=\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\right]\delta g^{\mu\nu}+\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\partial_\alpha\left(\delta g^{\mu\nu}\right). \tag{2.1.9}$$

Logo,

$$\delta\left(\mathcal{L}_M\sqrt{-g}\right)=\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial g^{\mu\nu}}\delta g^{\mu\nu}+\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\delta g^{\mu\nu}\right]-\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\right]\delta g^{\mu\nu}, \tag{2.1.10}$$

rearranjando os termos

$$\delta\left(\mathcal{L}_M\sqrt{-g}\right)=\left\{\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial g^{\mu\nu}}-\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\right]\right\}\delta g^{\mu\nu}+\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\delta g^{\mu\nu}\right]. \tag{2.1.11}$$

Assim, a segunda integral em (2.1.5) se torna

$$\begin{aligned}\int\delta\left(\sqrt{-g}\mathcal{L}_M\right)d^4x=\int\left\{\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial g^{\mu\nu}}-\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\right]\right\}\delta g^{\mu\nu}d^4x\\+\int\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\delta g^{\mu\nu}\right]d^4x,\end{aligned} \tag{2.1.12}$$

o termo entre colchetes é zero por se tratar de uma derivada total, já o termo entre chaves é a definição do *tensor energia-momento* $T_{\mu\nu}$[12]

$$T_{\mu\nu}=\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial g^{\mu\nu}}-\partial_\alpha\left[\frac{\partial\left(\mathcal{L}_M\sqrt{-g}\right)}{\partial\left(\partial_\alpha g^{\mu\nu}\right)}\right]. \tag{2.1.13}$$

Finalmente, pelo princípio de mínima ação,

$$\delta\mathcal{S}=\frac{1}{2\kappa}\int\left(R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}\right)\delta g^{\mu\nu}\sqrt{-g}d^4x-\frac{1}{2}\int T_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu}d^4x, \tag{2.1.14}$$

$$\delta\mathcal{S}=\int\left[R_{\mu\nu}-\frac{1}{2}Rg_{\mu\nu}-\kappa T_{\mu\nu}\right]\delta g^{\mu\nu}\sqrt{-g}d^4x=0 \tag{2.1.15}$$

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.1.16}$$

A eq. (2.1.16) é um conjunto de seis equações diferenciais para a métrica, pois a simetria nos índices reduz as dezesseis equações iniciais para um total de apenas dez e a *Identidade de Bianchi* reduz esse número em ainda mais quatro equações [9].

Em contrapartida às equações de Maxwell a equação tensorial acima é não-linear, essa conclusão é bem óbvia pelo fato de o escalar de Ricci ser composto por derivadas de segunda ordem da métrica. Aqui o campo gravitacional não tem origem no potencial newtoniano, a gravidade é a deformação na geometria de uma região do espaço-tempo devido a presença de um corpo massivo. Outra observação importante é que o termo à esquerda nas equaçãoes de campo de Einstein representam a geometria de uma determinada região do espaço-tempo, enquanto o termo à direita representa um termo de fonte, ou seja, a distribuição de matéria e energia nessa mesma região. Assim, podemos afirmar que o Einstein encontrou uma relação matemática que descreve como a geometria do espaço-tempo é afetada sob a presença de uma fonte de matéria.

Apesar de origens completamente distintas, essas duas teorias devem ser compatíveis com algumas considerações. Podemos fazer esta afirmação tendo em vista o sucesso que a gravitação newtoniana teve antes de Einstein. O chamado *limite newtoniano* é definido por meio de três condições: as partículas se movem com velocidades não relativísticas, o campo gravitacional é fraco (podendo ser considerado como uma pequena perturbação no espaço de Minkowski), e o campo é estático (não varia com o tempo). A métrica que resulta da solução das equações de Einstein nesse cenário é a seguinte [9]

$$g^{\mu\nu} = \begin{pmatrix} -1 - \frac{\phi}{c^2} & 0 & 0 & 0 \\ 0 & 1 - \frac{\phi}{c^2} & 0 & 0 \\ 0 & 0 & 1 - \frac{\phi}{c^2} & 0 \\ 0 & 0 & 0 & 1 - \frac{\phi}{c^2} \end{pmatrix}, \tag{2.1.17}$$

onde ϕ é o potencial newtoniano.

2.2 Cálculo geral do propagador em $(3+1)$ dimensões

Nesta seção vamos calcular a forma geral do propagador para teorias gravitacionais não-massivas. A ideia é retomar o procedimento visto no capítulo 2, mas para os operadores da base de operadores de Barnes-Rivers que em D dimensões é formada pelo conjunto a seguir [17]

$$P^{(1)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} \left(\theta_{\mu\kappa}\omega_{\nu\lambda} + \theta_{\mu\lambda}\omega_{\nu\kappa} + \theta_{\nu\kappa}\omega_{\mu\lambda} + \theta_{\nu\lambda}\omega_{\mu\kappa} \right), \tag{2.2.1}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda} = \frac{1}{2} \left(\theta_{\mu\kappa}\theta_{\nu\lambda} + \theta_{\mu\lambda}\theta_{\nu\kappa} \right) - \frac{1}{D-1} \theta_{\mu\nu}\theta_{\kappa\lambda}, \tag{2.2.2}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda} = \frac{1}{D-1} \theta_{\mu\nu}\theta_{\kappa\lambda}, \tag{2.2.3}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\nu}\omega_{\kappa\lambda}, \tag{2.2.4}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{D-1}} \left(\theta_{\mu\nu}\omega_{\kappa\lambda} + \theta_{\kappa\lambda}\omega_{\mu\nu} \right), \tag{2.2.5}$$

onde

$$\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}, \tag{2.2.6}$$

$$\omega_{\mu\nu} = \frac{p^\mu p^\nu}{p^2}. \tag{2.2.7}$$

No entanto, a nossa abordagem inicialmente ocorrerá num cenário em $(3+1)$ dimensões

$$P^{(1)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}\left(\theta_{\mu\kappa}\omega_{\nu\lambda} + \theta_{\mu\lambda}\omega_{\nu\kappa} + \theta_{\nu\kappa}\omega_{\mu\lambda} + \theta_{\nu\lambda}\omega_{\mu\kappa}\right), \tag{2.2.8}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}\left(\theta_{\mu\kappa}\theta_{\nu\lambda} + \theta_{\mu\lambda}\theta_{\nu\kappa}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda}, \tag{2.2.9}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda} = \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda}, \tag{2.2.10}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\nu}\omega_{\kappa\lambda}, \tag{2.2.11}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\omega_{\kappa\lambda} + \theta_{\kappa\lambda}\omega_{\mu\nu}\right). \tag{2.2.12}$$

As contrações dos projetores transversal e longitudinal estão contidas na tabela (2.1), a seguir vamos calcular algumas contrações e exibiremos a tabela completa.

	$\theta^{\alpha}{}_{\nu}$	$\omega^{\alpha}{}_{\nu}$
$\theta_{\mu\alpha}$	$\theta_{\mu\nu}$	0
$\omega_{\mu\alpha}$	0	$\omega_{\mu\nu}$

Tabela 2.1: Álgebra dos projetores transversal e longitudinal

$$\begin{aligned} P^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = {} & \frac{1}{2}\left(\theta_{\mu\kappa}\omega_{\nu\lambda} + \theta_{\mu\lambda}\omega_{\nu\kappa} + \theta_{\nu\kappa}\omega_{\mu\lambda} + \theta_{\nu\lambda}\omega_{\mu\kappa}\right) \\ & \times \frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right), \end{aligned} \tag{2.2.13}$$

$$\begin{aligned} P^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = {} & \frac{1}{4}\Big[\theta_{\mu\kappa}\omega_{\nu\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right) \\ & + \theta_{\mu\lambda}\omega_{\nu\kappa}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right) \\ & + \theta_{\nu\kappa}\omega_{\mu\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right) \\ & + \theta_{\nu\lambda}\omega_{\mu\kappa}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right)\Big], \end{aligned} \tag{2.2.14}$$

usando os resultados da tabela (2.1), temos

$$P^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\nu\beta}\omega_{\mu\alpha}\right) = P^{(1)}_{\mu\nu,\alpha\beta}.$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \left[\frac{1}{2}\left(\theta_{\mu\kappa}\theta_{\nu\lambda} + \theta_{\mu\lambda}\theta_{\nu\kappa}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda}\right] \times \left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3}\theta_{\alpha\beta}\theta^{\kappa\lambda}\right], \tag{2.2.15}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{4}\left(\theta_{\mu\kappa}\theta_{\nu\lambda}+\theta_{\mu\lambda}\theta_{\nu\kappa}\right)\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{6}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)\theta_{\mu\nu}\theta_{\kappa\lambda}$$
$$-\frac{1}{6}\left(\theta_{\mu\kappa}\theta_{\nu\lambda}+\theta_{\mu\lambda}\theta_{\nu\kappa}\right)\theta_{\alpha\beta}\theta^{\kappa\lambda}+\frac{1}{9}\theta_{\mu\nu}\theta_{\kappa\lambda}\theta_{\alpha\beta}\theta^{\kappa\lambda}, \quad (2.2.16)$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\nu\beta}+\theta_{\mu\beta}\theta_{\nu\alpha}\right)-\frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta} = P^{(2)}_{\mu\nu,\alpha\beta}. \quad (2.2.17)$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda}\times\frac{1}{3}\theta_{\alpha\beta}\theta^{\kappa\lambda} = \frac{3}{9}\theta_{\mu\nu}\theta_{\alpha\beta} = P^{(0-\theta)}_{\mu\nu,\alpha\beta}. \quad (2.2.18)$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \omega_{\mu\nu}\omega_{\kappa\lambda}\omega_{\alpha\beta}\omega^{\kappa\lambda} = \omega_{\mu\nu}\omega_{\alpha\beta} = P^{(0-\omega)}_{\mu\nu,\alpha\beta}. \quad (2.2.19)$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda}\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right) = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}. \quad (2.2.20)$$

Aqui devemos fazer um importante comentário. As contrações que envolvem os projetores $P^{(0-\theta)}_{\mu\nu,\alpha\beta}, P^{(0-\omega)}_{\mu\nu,\alpha\beta}$ e $P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}$ não é comutativa, ou seja, ao trocar-se a ordem o resultado muda. Observe:

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\omega_{\kappa\lambda}+\theta_{\kappa\lambda}\omega_{\mu\nu}\right)\times\frac{1}{3}\theta_{\alpha\beta}\theta^{\kappa\lambda} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\alpha\beta}. \quad (2.2.21)$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \omega_{\mu\nu}\omega_{\kappa\lambda}\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right) = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\alpha\beta}. \quad (2.2.22)$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\omega_{\kappa\lambda}+\theta_{\kappa\lambda}\omega_{\mu\nu}\right)\times\omega_{\alpha\beta}\omega^{\kappa\lambda} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}. \quad (2.2.23)$$

$$\begin{aligned} P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\omega_{\kappa\lambda}+\theta_{\kappa\lambda}\omega_{\mu\nu}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right), \\ &= \frac{1}{3}\theta_{\mu\nu}\omega_{\kappa\lambda}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right)+\frac{1}{3}\theta_{\kappa\lambda}\omega_{\mu\nu}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right), \\ &= \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}+\omega_{\mu\nu}\omega_{\alpha\beta} = P^{(0-\theta)}_{\mu\nu,\alpha\beta}+P^{(0-\omega)}_{\mu\nu,\alpha\beta}. \end{aligned} \quad (2.2.24)$$

Seguindo o método acima, o leitor pode verificar que calcular o restante das contrações é um processo relativamente cansativo, por isso sua omissão torna-se razoável. A tabela completa da álgebra dos projetores de Barnes-Rivers é mostrada a seguir Note que todos os resultados obtidos no cálculo dessas contrações podem ser escritos em termos dos operadores contidos no conjunto das eqs. (2.2.8) - (2.2.12), a razão disto reside no fato de este conjunto possuir todas as combinações lineares entre os projetores transversal e longitudinal que são linearmente independentes entre si. Isso quer dizer que (2.2.8) - (2.2.12) representam todas as combinações possíveis entre $\omega_{\mu\nu}$ e $\theta_{\mu\nu}$ não-redundantes.

Depois da apresentação da base a ser utilizada, seguiremos com o cálculo geral do propagador para teorias gravitacionais. O procedimento que iremos adotar consiste em escrever a lagrangiana da teoria na forma

$$\mathcal{L} = \frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta}, \quad (2.2.25)$$

	$P^{(1)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(2)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta}$
$P^{(1)}_{\mu\nu,\kappa\lambda}$	$P^{(1)}_{\mu\nu,\alpha\beta}$	0	0	0	0
$P^{(2)}_{\mu\nu,\kappa\lambda}$	0	$P^{(2)}_{\mu\nu,\alpha\beta}$	0	0	0
$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}$	0	0	$P^{(0-\theta)}_{\mu\nu,\alpha\beta}$	0	$\frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}$
$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}$	0	0	0	$P^{(0-\omega)}_{\mu\nu,\alpha\beta}$	$\frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\alpha\beta}$
$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}$	0	0	$\frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\alpha\beta}$	$\frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}$	$P^{(0-\theta)}_{\mu\nu,\alpha\beta}+P^{(0-\omega)}_{\mu\nu,\alpha\beta}$

Tabela 2.2: Álgebra dos operadores de Barnes-Rivers

que é conhecida como forma bilinear por apresentar o operador bilinear $\mathcal{O}_{\mu\nu,\alpha\beta}$. A forma mais geral para este operador é

$$\mathcal{O}_{\mu\nu,\alpha\beta} = a_1 P^{(1)} + a_2 P^{(2)} + a_3 P^{(0-\theta)} + a_4 P^{(0-\omega)} + a_5 P^{(0-\theta\omega)}, \tag{2.2.26}$$

cuja forma inversa pode ser escrita como

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = b_1 P^{(1)} + b_2 P^{(2)} + b_3 P^{(0-\theta)} + b_4 P^{(0-\omega)} + b_5 P^{(0-\theta\omega)}. \tag{2.2.27}$$

Aqui cabe enfatizar duas coisas importantes: o objetivo é escrever os coeficientes $b's$ em termos dos $a's$, e que o operador $\mathcal{O}_{\mu\nu,\alpha\beta}$ possui duas simetrias em seus índices, tais simetrias advém da própria estrutura da base de Barnes-Rivers. A vírgula que separa os índices tensoriais dos operadores em dois pares serve para indicar as simetrias mencionadas, ou seja, um projetor qualquer $X_{\mu\nu,\alpha\beta}$ possui uma simetria relacionada à permutação entre os pares separados

$$X_{\mu\nu,\alpha\beta} = X_{\alpha\beta,\mu\nu}, \tag{2.2.28}$$

e uma associada à permutação nos índices dos próprios pares

$$X_{\mu\nu,\alpha\beta} = X_{\nu\mu,\beta\alpha}. \tag{2.2.29}$$

Conhecendo-se a seguinte identidade que pode ser demonstrada por inspeção direta

$$\mathcal{I}_{\mu\nu,\alpha\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\nu\alpha}\eta_{\mu\beta}\right) = P^{(1)}_{\mu\nu,\alpha\beta} + P^{(2)}_{\mu\nu,\alpha\beta} + P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} \tag{2.2.30}$$

e usando a prescrição comum da literatura $\mathcal{O}\mathcal{O}^{-1} = \mathcal{I}$, temos

$$\begin{aligned}\mathcal{I}_{\mu\nu,\alpha\beta} = &\left(a_1 P^{(1)} + a_2 P^{(2)} + a_3 P^{(0-\theta)} + a_4 P^{(0-\omega)} + a_5 P^{(0-\theta\omega)}\right) \times \\ &\qquad\left(b_1 P^{(1)} + b_2 P^{(2)} + b_3 P^{(0-\theta)} + b_4 P^{(0-\omega)} + b_5 P^{(0-\theta\omega)}\right),\end{aligned}$$

desenvolvendo e usando os resultados da tabela (2.2)

$$\begin{aligned}P^{(1)}_{\mu\nu,\alpha\beta} + P^{(2)}_{\mu\nu,\alpha\beta} + P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} = &\; a_3\left(b_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}} b_5 P^{(\theta\omega)}_{\mu\nu,\alpha\beta}\right) \\ &+ a_1 b_1 P^{(1)}_{\mu\nu,\alpha\beta} + a_2 b_2 P^{(2)}_{\mu\nu,\alpha\beta} + a_4\left(b_4 P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}} b_5 P^{(\omega\theta)}_{\mu\nu,\alpha\beta}\right) \\ &+ a_5\left[\frac{1}{\sqrt{3}} b_3 P^{(\omega\theta)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}} b_4 P^{(\theta\omega)}_{\mu\nu,\alpha\beta} + b_5\left(P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta}\right)\right],\end{aligned}$$

onde, por simplicidade, escrevemos $P^{(\theta\omega)}_{\mu\nu,\alpha\beta} = \theta_{\mu\nu}\omega_{\alpha\beta}$ e $P^{(\omega\theta)}_{\mu\nu,\alpha\beta} = \omega_{\mu\nu}\theta_{\alpha\beta}$.

$$
\begin{aligned}
P^{(1)}_{\mu\nu,\alpha\beta} + P^{(2)}_{\mu\nu,\alpha\beta} + P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} = & \left(a_3b_3 + a_5b_5\right) P^{(0-\theta)}_{\mu\nu,\alpha\beta} \\
& + a_1b_1 P^{(1)}_{\mu\nu,\alpha\beta} + a_2b_2 P^{(2)}_{\mu\nu,\alpha\beta} + \left(a_4b_4 + a_5b_5\right) P^{(0-\omega)}_{\mu\nu,\alpha\beta} \\
& + \frac{1}{\sqrt{3}}\left(a_3b_5 + a_5b_4\right) P^{(\theta\omega)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}}\left(a_4b_5 + a_5b_3\right) P^{(\omega\theta)}_{\mu\nu,\alpha\beta},
\end{aligned}
$$

comparando os coeficientes, obtemos o seguinte sistema:

$$a_1b_1 = 1, \tag{2.2.31}$$

$$a_2b_2 = 1, \tag{2.2.32}$$

$$a_3b_3 + a_5b_5 = 1, \tag{2.2.33}$$

$$a_4b_4 + a_5b_5 = 1, \tag{2.2.34}$$

$$a_3b_5 + a_5b_4 = 0, \tag{2.2.35}$$

$$a_4b_5 + a_5b_3 = 0. \tag{2.2.36}$$

Este sistema não possui solução direta, vamos escrevê-lo na sua forma matricial e, por meio do escalonamento, verificar se uma das equações é combinação linear das outras

$$
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & a_3 & 0 & a_5 & 1 \\
0 & 0 & 0 & a_4 & a_5 & 1 \\
0 & 0 & 0 & a_5 & a_3 & 0 \\
0 & 0 & a_5 & 0 & a_4 & 0
\end{pmatrix}. \tag{2.2.37}
$$

Multiplicando a linha 6 por a_3, a linha 3 por $-a_5$ e substituindo a linha 6 pela soma das novas linha 3 e 6

$$
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & a_3 & 0 & a_5 & 1 \\
0 & 0 & 0 & a_4 & a_5 & 1 \\
0 & 0 & 0 & a_5 & a_3 & 0 \\
0 & 0 & a_5 & 0 & a_4 & 0
\end{pmatrix}
\sim
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & -a_3a_5 & 0 & -a_5^2 & -a_5 \\
0 & 0 & 0 & a_4 & a_5 & 1 \\
0 & 0 & 0 & a_5 & a_3 & 0 \\
0 & 0 & 0 & 0 & a_3a_4 - a_5^2 & -a_5
\end{pmatrix}, \tag{2.2.38}
$$

multiplicando a linha 5 por $-a_4$, a linha 4 por a_5 e dividindo a linha 3 por $-a_5$

$$
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & -a_3a_5 & 0 & -a_5^2 & -a_5 \\
0 & 0 & 0 & a_4 & a_5 & 1 \\
0 & 0 & 0 & a_5 & a_3 & 0 \\
0 & 0 & 0 & 0 & a_3a_4 - a_5^2 & -a_5
\end{pmatrix}
\sim
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & a_3 & 0 & a_5 & 1 \\
0 & 0 & 0 & a_4a_5 & a_5^2 & a_5 \\
0 & 0 & 0 & -a_4a_5 & -a_3a_4 & 0 \\
0 & 0 & 0 & 0 & a_3a_4 - a_5^2 & -a_5
\end{pmatrix}, \tag{2.2.39}
$$

substituímos a linha 5 pela soma das linhas 4 e 5 e dividimos a linha 4 por a_5

$$
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & a_3 & 0 & a_5 & 1 \\
0 & 0 & 0 & a_4a_5 & a_5^2 & a_5 \\
0 & 0 & 0 & -a_4a_5 & -a_3a_4 & 0 \\
0 & 0 & 0 & 0 & a_3a_4 - a_5^2 & -a_5
\end{pmatrix}
\sim
\begin{pmatrix}
a_1 & 0 & 0 & 0 & 0 & 1 \\
0 & a_2 & 0 & 0 & 0 & 1 \\
0 & 0 & a_3 & 0 & a_5 & 1 \\
0 & 0 & 0 & a_4 & a_5 & 1 \\
0 & 0 & 0 & 0 & a_5^2 - a_3a_4 & a_5 \\
0 & 0 & 0 & 0 & a_3a_4 - a_5^2 & -a_5
\end{pmatrix}. \tag{2.2.40}
$$

Finalmente, substituímos a linha 6 pela soma das linhas 5 e 6

$$\begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_3 & 0 & a_5 & 1 \\ 0 & 0 & 0 & a_4 & a_5 & 1 \\ 0 & 0 & 0 & 0 & a_5^2 - a_3a_4 & a_5 \\ 0 & 0 & 0 & 0 & a_3a_4 - a_5^2 & -a_5 \end{pmatrix} \sim \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 1 \\ 0 & a_2 & 0 & 0 & 0 & 1 \\ 0 & 0 & a_3 & 0 & a_5 & 1 \\ 0 & 0 & 0 & a_4 & a_5 & 1 \\ 0 & 0 & 0 & 0 & a_5^2 - a_3a_4 & a_5 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{2.2.41}$$

Desde que $a_1 \neq 0$, $a_2 \neq 0$ e $a_5^2 - a_3a_4 \neq 0$, obtemos um novo sistema

$$a_1b_1 = 1, \tag{2.2.42}$$

$$a_2b_2 = 1, \tag{2.2.43}$$

$$a_3b_3 + a_5b_5 = 1, \tag{2.2.44}$$

$$a_4b_4 + a_5b_5 = 1, \tag{2.2.45}$$

$$\left(a_5^2 - a_3a_4\right) b_5 = a_5. \tag{2.2.46}$$

Observe que uma das linhas da matriz representa uma equação do sistema e que uma dessas linhas foi zerada, isso significa que uma das equações poderia ser escrita como combinação linear das outras. O processo de aplicar operações elementares em cada uma das linhas dessa matriz, além de garantir que a solução a ser obtida para o novo sistema é exatamente igual à anterior, só conduz a linhas nulas se houver ao menos uma linha redundante. As primeira, segunda e quinta equações possuem solução direta

$$b_1 = \frac{1}{a_1}, \quad b_2 = \frac{1}{a_2}, \quad b_5 = \frac{a_5}{a_5^2 - a_3a_4}. \tag{2.2.47}$$

As duas restantes podem ser resolvidas facilmente, uma vez conhecido o valor de b_5, a terceira equação torna-se

$$a_3b_3 + a_5\frac{a_5}{a_5^2 - a_3a_4} = 1 \longrightarrow b_3 = -\frac{a_4}{a_5^2 - a_3a_4}. \tag{2.2.48}$$

A quarta

$$a_4b_4 + a_5\frac{a_5}{a_5^2 - a_3a_4} = 1 \longrightarrow b_4 = -\frac{a_3}{a_5^2 - a_3a_4}. \tag{2.2.49}$$

Assim, a solução final é

$$b_1 = \frac{1}{a_1}, \quad b_2 = \frac{1}{a_2}, \quad b_3 = -\frac{a_4}{a_5^2 - a_3a_4}, \tag{2.2.50}$$

$$b_4 = -\frac{a_3}{a_5^2 - a_3a_4}, \quad b_5 = \frac{a_5}{a_5^2 - a_3a_4}, \tag{2.2.51}$$

e o propagador assume a forma

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = \frac{1}{a_1}P^{(1)} + \frac{1}{a_2}P^{(2)} - \frac{a_4}{a_5^2 - a_3a_4}P^{(0-\theta)} - \frac{a_3}{a_5^2 - a_3a_4}P^{(0-\omega)} + \frac{a_5}{a_5^2 - a_3a_4}P^{(0-\theta\omega)},$$

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = \frac{1}{a_1}P^{(1)} + \frac{1}{a_2}P^{(2)} - \frac{1}{a_5^2 - a_3a_4}\left(a_4P^{(0-\theta)} + a_3P^{(0-\omega)} - a_5P^{(0-\theta\omega)}\right), \tag{2.2.52}$$

ou na forma de Feynman $\triangle_{\mu\nu,\alpha\beta} = i\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}$

$$\triangle_{\mu\nu,\alpha\beta} = i\left[\frac{1}{a_1}P^{(1)}_{\mu\nu,\alpha\beta} + \frac{1}{a_2}P^{(2)}_{\mu\nu,\alpha\beta} - \frac{1}{a_5^2 - a_3a_4}\left(a_4P^{(0-\theta)}_{\mu\nu,\alpha\beta} + a_3P^{(0-\omega)}_{\mu\nu,\alpha\beta} - a_5P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right)\right].$$

Em resumo, o cálculo do propagador para teorias gravitacionais terá como regra o seguinte roteiro:

1. Escrever a densidade lagrangiana na forma linearizada por meio da prescrição

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}. \tag{2.2.53}$$

2. Obter a forma bilinear da lagrangiana linearizada, ou seja, reescrevê-la sob a forma

$$\mathcal{L} = \frac{1}{2} h^{\mu\nu} \mathcal{O}_{\mu\nu,\alpha\beta} h^{\alpha\beta}. \tag{2.2.54}$$

3. Reescrever o operador $\mathcal{O}_{\mu\nu,\alpha\beta}$ em termos dos operadores da base de Barnes-Rivers.

4. Identificar os coeficientes $a_1,\ a_2,\ a_3,\ a_4$ e a_5 e substituir na eq. (2.2.27).

2.3 Propagador para o gráviton livre

O propagador mais simples é o do gráviton livre. De imediato podemos inferir que só haverá um polo no propagador, sendo este equivalente à relação de dispersão que se obtém para ondas gravitacionais planas, ou seja, só existe um modo de propagação da interação gravitacional neste cenário. Mais adiante, veremos que em outros quadros o propagador apresentará mais de um polo e estes estarão associados às constantes de acoplamento dos termos introduzidos.

A densidade lagrangiana adequada para a obtenção do propagador do gráviton livre é

$$\mathcal{L} = \mathcal{L}_{E-H} + \mathcal{L}_{G-F}, \tag{2.3.1}$$

onde $\mathcal{L}_{E-H}$ é originária da ação de Einstein-Hilbert, cuja forma explícita é

$$\mathcal{S}_{E-H} = \int \sqrt{-g} R dx^4, \tag{2.3.2}$$

e $\mathcal{L}_{G-F}$ é o termo de fixação de calibre (gauge-fixing)

$$\mathcal{L}_{G-F} = \frac{1}{2\alpha} \left(\partial_\nu h^{\mu\nu} - \frac{1}{2} \partial^\mu h \right) \left(\partial^\kappa h_{\mu\kappa} - \frac{1}{2} \partial_\mu h \right), \tag{2.3.3}$$

sendo α uma constante adimensional. Existem diversos tipos de gauge na literatura [17], o que adotamos neste trabalho recebe o nome de *de Donder*. A introdução de um termo de gauge-fixing é aqui necessária, pois $\mathcal{L}_{E-H}$ possui uma simetria de gauge local originária da linearização (2.2.53). De fato, a lagrangiana (2.3.6) é invariante sob a transformação de coordenadas infinitesimal

$$x^\mu \to x^\mu + \xi^\mu (x), \tag{2.3.4}$$

onde $\xi^\mu (x)$ é um vetor infinitesimal que depende de x^μ. Com isso, o escalar de Ricci, que compõe a ação (2.3.2) é invariante sob a seguinte transformação no campo $h_{\mu\nu}$,

$$h_{\mu\nu} (x) \to h_{\mu\nu} (x) - \partial_\nu \xi_\mu (x) - \partial_\mu \xi_\nu (x). \tag{2.3.5}$$

conhecida como transformação de gauge. Essa transformação não altera a física do campo gravitacional que é representado pelo tensor $h_{\mu\nu}$, ou seja, independente da forma do vetor $\xi_\mu (x)$, o campo gravitacional será exatamente o mesmo. Assim, temos uma liberdade de gauge sobre o campo $h_{\mu\nu}$ que pode ser interpretada como uma livre escolha do campo $h_{\mu\nu}$ dado pela expressão (2.3.5).

Seguindo os passos descritos na seção anterior, iniciaremos escrevendo a expansão da lagrangiana de Einstein-Hilbert em segunda ordem (veja o apêndice B)

$$\mathcal{L}_{E-H} = -\frac{1}{4}\partial^\alpha h\partial_\alpha h + \frac{1}{2}\partial^\alpha h\partial^\beta h_{\alpha\beta} + \frac{1}{4}\partial^\gamma h_{\alpha\beta}\partial_\gamma h^{\alpha\beta} - \frac{1}{2}\partial^\beta h^{\alpha\gamma}\partial_\alpha h_{\beta\gamma}. \tag{2.3.6}$$

No caso específico do escalar de Ricci é necessário os termos de segunda ordem, a justificativa está no fato de os termos de primeria ordem serem derivadas totais e não possuírem uma forma quadrática no campo $h^{\mu\nu}$, requisito imprescindível para obtermos sua forma bilinear. Aplicando a regra do produto de derivação podemos reescrever as parcelas de forma conveniente

$$\partial^\alpha\left(h\partial_\alpha h\right) = \partial^\alpha h\partial_\alpha h + h\Box h, \tag{2.3.7}$$
$$\partial^\alpha\left(h\partial^\beta h_{\alpha\beta}\right) = \partial^\alpha h\partial^\beta h_{\alpha\beta} + h\partial^\alpha\partial^\beta h_{\alpha\beta}, \tag{2.3.8}$$
$$\partial^\gamma\left(h_{\alpha\beta}\partial_\gamma h^{\alpha\beta}\right) = \partial^\gamma h_{\alpha\beta}\partial_\gamma h^{\alpha\beta} + h_{\alpha\beta}\Box h^{\alpha\beta}, \tag{2.3.9}$$
$$\partial^\beta\left(h^{\alpha\gamma}\partial_\alpha h_{\beta\gamma}\right) = \partial^\beta h^{\alpha\gamma}\partial_\alpha h_{\beta\gamma} + h^{\alpha\gamma}\partial^\beta\partial_\alpha h_{\beta\gamma}. \tag{2.3.10}$$

Observe que podemos substituir diretamente esses termos, pois as parcelas que possuem derivadas totais não irão contribuir. Logo,

$$\mathcal{L}_{E-H} = \frac{1}{4}h\Box h - \frac{1}{2}h\partial^\alpha\partial^\beta h_{\alpha\beta} - \frac{1}{4}h_{\alpha\beta}\Box h^{\alpha\beta} + \frac{1}{2}h^{\alpha\gamma}\partial^\beta\partial_\alpha h_{\beta\gamma}. \tag{2.3.11}$$

Vamos agora usar a métrica para expressarmos a lagrangiana linearizada na sua forma bilinear

$$\mathcal{L}_{E-H} = \frac{1}{2}h^{\mu\nu}\left(\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\partial_\alpha\partial_\beta - \frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\Box + \eta_{\nu\alpha}\partial_\beta\partial_\mu\right)h^{\alpha\beta}. \tag{2.3.12}$$

Trabalhando com a parte correspondente ao termo de fixação de calibre, no intuito de também reescrevê-la na forma bilinear

$$\begin{aligned}\mathcal{L}_{G-F} &= \frac{1}{2\alpha}\left(\partial_\nu h^{\mu\nu} - \frac{1}{2}\partial^\mu h\right)\times\left(\partial^\kappa h_{\mu\kappa} - \frac{1}{2}\partial_\mu h\right),\\ &= \frac{1}{2\alpha}\left(\partial_\nu h^{\mu\nu}\partial^\kappa h_{\mu\kappa} - \frac{1}{2}\partial_\nu h^{\mu\nu}\partial_\mu h - \frac{1}{2}\partial^\mu h\partial^\kappa h_{\mu\kappa} + \frac{1}{4}\partial^\mu h\partial_\mu h\right).\end{aligned} \tag{2.3.13}$$

Por meio da renomeação de índices contraídos da terceira parcela entre parênteses, temos

$$\mathcal{L}_{G-F} = \frac{1}{2\alpha}\left(\partial_\nu h^{\mu\nu}\partial^\kappa h_{\mu\kappa} - \partial_\nu h^{\mu\nu}\partial_\mu h + \frac{1}{4}\partial^\mu h\partial_\mu h\right). \tag{2.3.14}$$

Novamente, reescrevendo as parcelas em termos de derivadas do produto e desprezando os termos relacionados a derivadas totais, resulta

$$\mathcal{L}_{G-F} = \frac{1}{2\alpha}\left(h\partial_\mu\partial_\nu h^{\mu\nu} - h^{\mu\nu}\partial_\nu\partial^\kappa h_{\mu\kappa} - \frac{1}{4}h\Box h\right), \tag{2.3.15}$$

usamos a métrica para expor a sua forma bilinear

$$\mathcal{L}_{G-F} = \frac{1}{2\alpha}h^{\mu\nu}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\nu\beta}\partial_\mu\partial_\alpha - \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box\right)h^{\alpha\beta}. \tag{2.3.16}$$

A lagrangiana total $\mathcal{L} = \mathcal{L}_{E-H} + \mathcal{L}_{G-F}$, na sua forma linearizada é a seguinte

$$\mathcal{L} = \frac{1}{2}h^{\mu\nu}\left[\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\partial_\alpha\partial_\beta - \frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\Box + \eta_{\nu\alpha}\partial_\beta\partial_\mu \right.$$
$$\left. + \frac{1}{\alpha}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\nu\beta}\partial_\mu\partial_\alpha - \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box\right)\right]h^{\alpha\beta}, \quad (2.3.17)$$

de onde extraímos o operador

$$\mathcal{O}_{\mu\nu,\alpha\beta} = \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\partial_\alpha\partial_\beta - \frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\Box + \eta_{\nu\alpha}\partial_\mu\partial_\beta$$
$$+ \frac{1}{\alpha}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\nu\beta}\partial_\mu\partial_\alpha - \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box\right), \quad (2.3.18)$$

que constitui o núcleo estrutural da lagrangiana

$$\mathcal{L} = \frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta}. \quad (2.3.19)$$

2.3.1 Simetrização do operador $\mathcal{O}$ e cálculo do propagador

Observando atentamente o operador da eq. (2.3.18), podemos observar que existem parcelas que não satisfazem às simetrias presentes nos projetores da base de Barnes-Rivers, eliminando a possibilidade de serem escritos em termos dos mesmos. Neste caso, torna-se imprescindível a simetrização de cada uma das parcelas não-simetrizadas, tal simetrização pode ser realizada facilmente ao usarmos o fato de que as componentes simétricas de um tensor podem ser escritas como

$$A_{(\mu\nu)} = \frac{1}{2}\left(A_{\mu\nu} + A_{\nu\mu}\right). \quad (2.3.20)$$

Nos valemos deste resultado sustentados pela possibilidade de qualquer tensor ser dividido em duas partes: uma simétrica e outra antissimétrica e também pela condição de o operador $\mathcal{O}$ ser bilinear no campo $h^{\mu\nu}$, esta segunda afirmativa anula a componente antissimétrica que decorre da primeira. Assim, quaisquer termos que não possuem simetria definida podem ser simetrizadas por meio da eq. (2.3.20). Deve-se ter bastante cuidado ao aplicar esse método, antes de tudo é necessário observar se a parcela a ser simetrizada já não possui alguma simetria como em $\eta_{\mu\nu}\partial_\alpha\partial_\beta$, os índices da métrica e as derivadas parciais comutam, logo há apenas uma simetrização a ser feita

$$\eta_{\mu\nu}\partial_\alpha\partial_\beta = \frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right), \quad (2.3.21)$$

em seguida, temos o produto de duas métricas $\eta_{\mu\alpha}\eta_{\nu\beta}$, já existe uma simetria presente

$$\eta_{\mu\alpha}\eta_{\nu\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right), \quad (2.3.22)$$

a última $\eta_{\nu\alpha}\partial_\mu\partial_\beta$ não possui qualquer simetria a ser aproveitada tornando necessário aplicar o método de simetrização duas vezes

$$\begin{aligned}\eta_{\nu\alpha}\partial_\mu\partial_\beta &= \frac{1}{2}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha\right),\\ &= \frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right),\\ &= \eta_{\nu\beta}\partial_\mu\partial_\alpha. \end{aligned} \quad (2.3.23)$$

Desse modo, os termos simetrizados correspondentes serão

$$\eta_{\mu\nu}\partial_\alpha\partial_\beta = \frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right), \tag{2.3.24}$$

$$\eta_{\mu\alpha}\eta_{\nu\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right), \tag{2.3.25}$$

$$\eta_{\nu\alpha}\partial_\mu\partial_\beta = \frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right), \tag{2.3.26}$$

ou ainda no espaço dos momentos

$$\eta_{\mu\nu}p_\alpha p_\beta = \frac{1}{2}\left(\eta_{\mu\nu}p_\alpha p_\beta + \eta_{\alpha\beta}p_\mu p_\nu\right), \tag{2.3.27}$$

$$\eta_{\mu\alpha}\eta_{\nu\beta} = \frac{1}{2}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right), \tag{2.3.28}$$

$$\eta_{\nu\alpha}p_\mu p_\beta = \frac{1}{4}\left(\eta_{\nu\alpha}p_\mu p_\beta + \eta_{\mu\alpha}p_\nu p_\beta + \eta_{\nu\beta}p_\mu p_\alpha + \eta_{\mu\beta}p_\nu p_\alpha\right), \tag{2.3.29}$$

Substituindo todas essas parcelas, agora simetrizadas, na eq. (2.3.18)

$$\begin{aligned}\mathcal{O}_{\mu\nu,\alpha\beta} = {} & \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right) - \frac{1}{4}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right)\Box \\ & + \frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right) + \frac{1}{\alpha}\left[\frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right)\right. \\ & \left. - \frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right) - \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box\right].\end{aligned} \tag{2.3.30}$$

Agora devemos usar as identidades dos operadores de Barnes-Rivers [17] listadas abaixo

$$\eta_{\mu\nu}\eta_{\alpha\beta} = 3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}, \tag{2.3.31}$$

$$\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu = 2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}, \tag{2.3.32}$$

$$\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha = 2P^{(1)} + 4P^{(0-\omega)}, \tag{2.3.33}$$

$$\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha} = 2\left(P^{(0-\theta)} + P^{(0-\omega)} + P^{(1)} + P^{(2)}\right), \tag{2.3.34}$$

para então escrever:

$$\begin{aligned}\mathcal{O}_{\mu\nu,\alpha\beta} = {} & \frac{1}{2}\left(3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box \\ & - \frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box - \frac{1}{2}\left(P^{(0-\theta)} + P^{(0-\omega)} + P^{(1)} + P^{(2)}\right)\Box \\ & + \frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right)\Box + \frac{1}{\alpha}\left[\frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\right. \\ & \left. - \frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right) - \frac{1}{4}\left(3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\right]\Box,\end{aligned} \tag{2.3.35}$$

agrupando os coeficientes semelhantes

$$\mathcal{O}_{\mu\nu,\alpha\beta} = \left(\frac{4\alpha - 3}{4\alpha}\right)\Box P^{(0-\theta)} - \frac{\Box}{2\alpha}P^{(1)} - \frac{\Box}{2}P^{(2)} - \frac{\Box}{4\alpha}P^{(0-\omega)} + \frac{\sqrt{3}\Box}{4\alpha}P^{(0-\theta\omega)}. \tag{2.3.36}$$

Com isso, é possível identificar os coeficientes

$$a_1 = -\frac{\Box}{2\alpha}, \quad a_2 = -\frac{\Box}{2}, \quad a_3 = \left(\frac{4\alpha - 3}{4\alpha}\right)\Box, \quad a_4 = -\frac{\Box}{4\alpha} \text{ e } \quad a_5 = \frac{\sqrt{3}\Box}{4\alpha}$$

necessários para escrevermos o propagador do gráviton livre por meio da eq. (2.2.52)

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = -\frac{2\alpha}{\Box}P^{(1)} - \frac{2}{\Box}P^{(2)} - \frac{4\alpha}{\Box}\left(-\frac{1}{4\alpha}P^{(0-\theta)} + \frac{4\alpha-3}{4\alpha}P^{(0-\omega)} - \frac{\sqrt{3}}{4\alpha}P^{(0-\theta\omega)}\right),$$

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = -\frac{2\alpha}{\Box}P^{(1)} - \frac{2}{\Box}P^{(2)} + \frac{P^{(0-\theta)} - (4\alpha-3)\,P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}}{\Box}. \tag{2.3.37}$$

Uma outra forma de escrever o propagador acima é usar a notação de Feynman

$$\Delta_{\mu\nu,\alpha\beta} = i\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}, \tag{2.3.38}$$

onde $\Delta_{\mu\nu,\alpha\beta}$ é o valor esperado no vácuo do produto temporalmente ordenado dos campos, resultado presente no estudo de Teoria Quântica de Campos (TQC)

$$\Delta_{\mu\nu,\alpha\beta}(\mathbf{x}-\mathbf{y}) = \langle 0\,|T\left[h_{\mu\nu}(\mathbf{x})\,h_{\alpha\beta}(\mathbf{y})\right]|\,0\rangle\,. \tag{2.3.39}$$

Assim, no espaço dos momentos, o propagador da teoria gravitacional de Einstein-Hilbert será

$$\Delta_{\mu\nu,\alpha\beta} = \frac{i}{p^2}\left[2\alpha P^{(1)}_{\mu\nu,\alpha\beta} + 2P^{(2)}_{\mu\nu,\alpha\beta} + (4\alpha-3)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} - P^{(0-\theta)}_{\mu\nu,\alpha\beta} - \sqrt{3}P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right]. \tag{2.3.40}$$

Note que o único polo do propagador é

$$p^2 = 0,$$

cuja relação de dispersão associada é

$$p^2 = p_0^2 - |\mathbf{p}|^2 = 0. \tag{2.3.41}$$

A velocidade de grupo desse modo de propagação pode ser obtida por meio da relação

$$u_g = \frac{dp_0}{d\,|\mathbf{p}|}, \tag{2.3.42}$$

logo,

$$u_g = 1, \tag{2.3.43}$$

que coincide com a velocidade de fase u_f

$$u_f = \frac{p_0}{|\mathbf{p}|} = 1. \tag{2.3.44}$$

Note que a propagação de sinais na teoria gravitacional de Einstein-Hilbert é causal. Nessa teoria o gráviton, partícula mediadora da interação gravitacional, possui velocidade de propagação igual a do fóton, que é a partícula mediadora da interação eletromagnética.

2.4 Unitariedade tree-level

A gravitação de Einstein-Hilbert, por ser uma teoria clássica de campos, precisa exibir propagação de sinais consistente com o princípio de causalidade e com a unitariedade. Logo, o modo não-massivo $p^2 = 0$ dos grávitons não deverá produzir excitações de norma negativa. Usaremos esta afirmação para encontrar um resultado que servirá de baliza para as próximas análises de unitariedade, por meio do mecanismo de saturação do propagador por correntes externas conservadas ($p_\mu J^{\mu\nu} = 0$). A análise dessa seção será voltada para as excitações usuais do gráviton (descritas no bojo da teoria de Einstein-Hilbert). Iniciamos nosso estudo apresentando o propagador saturado e discutindo quais projetores contribuem de fato para a saturação. Com isso, a saturação do propagador é dada por:

$$SP = J^{\mu\nu} Res \left[\Delta_{\mu\nu,\alpha\beta}\right] J^{\alpha\beta}. \tag{2.4.1}$$

Usando o propagador (2.3.40), temos

$$SP = J^{\mu\nu} Res \left[\frac{2i\alpha}{p^2} P^{(1)} + \frac{2i}{p^2} P^{(2)} + i\frac{(4\alpha - 3) P^{(0-\omega)} - P^{(0-\theta)} - \sqrt{3} P^{(0-\theta\omega)}}{p^2}\right] J^{\alpha\beta}. \tag{2.4.2}$$

Ao observarmos os operadores da base de Barnes-Rivers, podemos notar que todos os que são proporcionais ao projetor longitudinal $\omega_{\mu\nu}$ gerarão contribuição nula em razão da conservação da corrente, ou seja

$$J^{\mu\nu} P^{(1)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} = \frac{1}{2} J^{\mu\nu} \left(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\nu\beta}\omega_{\mu\alpha}\right) J^{\alpha\beta} = 0, \tag{2.4.3}$$

$$J^{\mu\nu} P^{(0-\omega)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} = J^{\mu\nu} \omega_{\mu\nu}\omega_{\alpha\beta} J^{\alpha\beta} = 0, \tag{2.4.4}$$

$$J^{\mu\nu} P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} = J^{\mu\nu} \left(\theta_{\mu\nu}\omega_{\alpha\beta} + \theta_{\alpha\beta}\omega_{\mu\nu}\right) J^{\alpha\beta} = 0. \tag{2.4.5}$$

Apenas dois contribuirão, são estes $P^{(2)}_{\mu\nu,\alpha\beta}$ e $P^{(0-\theta)}_{\mu\nu,\alpha\beta}$. O primeiro resulta ser

$$\begin{aligned} J^{\mu\nu} P^{(2)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} &= \frac{1}{2} J^{\mu\nu} \left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}\right) J^{\alpha\beta} - \frac{1}{3} J^{\mu\nu}\theta_{\mu\nu}\theta_{\alpha\beta} J^{\alpha\beta}, \\ &= J_{\alpha\beta} J^{\alpha\beta} - \frac{1}{3} \left(J^{\alpha}{}_{\alpha}\right)^2. \end{aligned} \tag{2.4.6}$$

Note que a contração em $\theta_{\mu\nu}$ funciona como levantamento/abaixamento de índices, uma vez que $\theta_{\mu\nu} = \eta_{\mu\nu} - \omega_{\mu\nu}$, enquanto a contribuição advinda de $\omega_{\mu\nu}$ se anula. O segundo foi calculado acima, logo

$$J^{\mu\nu} P^{(0-\theta)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} = \frac{1}{3} \left(J^{\alpha}{}_{\alpha}\right)^2. \tag{2.4.7}$$

Considerando apenas os termos não nulos, obtemos a saturação:

$$\begin{aligned} SP &= i Res \left[\frac{1}{p^2}\left(2J_{\alpha\beta}J^{\alpha\beta} - \frac{2}{3}\left(J^{\alpha}{}_{\alpha}\right)^2 - \frac{1}{3}\left(J^{\alpha}{}_{\alpha}\right)^2\right)\right], \\ &= i Res \left[\frac{1}{p^2}\left(2J_{\alpha\beta}J^{\alpha\beta} - \left(J^{\alpha}{}_{\alpha}\right)^2\right)\right]. \end{aligned} \tag{2.4.8}$$

A partir da conservação da corrente, podemos derivar as seguintes relações

$$J_{00} = \frac{p_a p_c}{p_0^2} J_{ca}, \tag{2.4.9}$$

$$J_{0a} = \frac{p_c}{p_0} J_{ca}. \tag{2.4.10}$$

que nos permitem reescrever a $2J_{\alpha\beta}J^{\alpha\beta} - (J^{\alpha}{}_{\alpha})^2$ como se segue

$$2J_{\alpha\beta}J^{\alpha\beta} - (J^{\alpha}{}_{\alpha})^2 = 2J_{0\beta}J^{0\beta} + 2J_{c\beta}J^{c\beta} - \left(J^0{}_0 + J^a{}_a\right)^2, \tag{2.4.11}$$

$$= 2J_{00}J^{00} + 2J_{0a}J^{0a} + 2J_{c0}J^{c0} + 2J_{ca}J^{ca} - (J_{00} - J_{aa})^2. \tag{2.4.12}$$

Independentemente da assinatura usada para a métrica, teremos $J_{00} = J^{00}$, $J_{0a} = -J^{0a}$, $J_{ca} = J^{ca}$. Assim, temos:

$$2J_{\alpha\beta}J^{\alpha\beta} - (J^{\alpha}{}_{\alpha})^2 = 2(J_{00})^2 - (J_{00})^2 + 2J_{00}J_{aa} - (J_{aa})^2 - 2(J_{0a})^2 - 2(J_{c0})^2 + 2(J_{ca})^2. \tag{2.4.13}$$

Usando as eqs. (2.4.9), temos

$$2J_{\alpha\beta}J^{\alpha\beta} - (J^{\alpha}{}_{\alpha})^2 = \frac{(p_a p_c J_{ca})^2}{p_0^4} + 2\frac{p_a p_c}{p_0^2}J_{ca}J_{aa} - (J_{aa})^2 - 2\frac{(p_c J_{ca})^2}{p_0} - 2\frac{(p_a J_{ca})^2}{p_0} + 2(J_{ca})^2, \tag{2.4.14}$$

agrupando os termos semelhantes, resulta

$$2J_{\alpha\beta}J^{\alpha\beta} - (J^{\alpha}{}_{\alpha})^2 = \left(\frac{p_a p_c J_{ca}}{p_0^2} + J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{p_0} - 2(J_{aa})^2. \tag{2.4.15}$$

Logo, o resíduo em $p^2 = 0$ será dado por

$$Res|_{p_0^2 - |\mathbf{p}|^2 = 0} = \left(\frac{p_a p_c J_{ca}}{p_0^2} + J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{p_0} - 2(J_{aa})^2,$$
$$= \left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2} + J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{|\mathbf{p}|} - 2(J_{aa})^2,$$

e a saturação do propagador

$$SP = i\left[\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2} + J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{|\mathbf{p}|} - 2(J_{aa})^2\right]. \tag{2.4.16}$$

A informação que deve ser ressaltada é a seguinte: para que a gravitação de Einstein-Hilbert seja unitária, a parte imaginária do resíduo da saturação deve ser positiva. Assim, a expressão acima, deve ser também positiva. Portanto, decorre:

$$\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2} + J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{|\mathbf{p}|} - 2(J_{aa})^2 > 0. \tag{2.4.17}$$

2.5 Graus de liberdade (abordagem 1)

Nesta seção vamos discutir sobre os graus de liberdade do campo gravitacional, dentro da teoria de Einstein-Hilbert. Seguiremos um procedimento similar ao adotado na Ref. [22] para efetuar essa análise e chegaremos a conclusão de que essa abordagem não é adequada em se tratando de teorias invariantes de calibre.

Note que o campo $h^{\mu\nu}$ possui em princípio um total de 16 componentes independentes, mas, a existência da simetria entre μ e ν reduz esse número para apenas 10. A equação de movimento obtida da lagrangiana (2.2.54) é dada a seguir

$$\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta} = 0, \tag{2.5.1}$$

onde o operador $\mathcal{O}$ é o mesmo da eq. (2.3.36). Iniciamos por contrair a eq. (2.5.1) com $p^\mu p^\nu$, ou seja,

$$p^\mu p^\nu \mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta} = 0. \tag{2.5.2}$$

A forma do operador $\mathcal{O}$ está presente na eq. (2.2.26), assim

$$p^\mu p^\nu \left[a_1 P^{(1)}_{\mu\nu,\alpha\beta} + a_2 P^{(2)}_{\mu\nu,\alpha\beta} + a_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta} + a_4 P^{(0-\omega)}_{\mu\nu,\alpha\beta} + a_5 P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right] h^{\alpha\beta} = 0. \tag{2.5.3}$$

Portanto, para obter o resultado da equação anterior, devemos contrair os projetores (2.2.8)-(2.2.12) com $p^\mu p^\nu$ individualmente:

$$p^\mu p^\nu P^{(0-\theta)}_{\mu\nu,\alpha\beta} = \frac{1}{3}p^\mu p^\nu \theta_{\mu\nu}\theta_{\alpha\beta} = 0, \tag{2.5.4}$$

$$p^\mu p^\nu P^{(1)}_{\mu\nu,\alpha\beta} = \frac{1}{2}p^\mu p^\nu \left(\theta_{\mu\alpha}\omega_{\nu\beta} + \theta_{\mu\beta}\omega_{\nu\alpha} + \theta_{\nu\alpha}\omega_{\mu\beta} + \theta_{\nu\beta}\omega_{\mu\alpha}\right) = 0, \tag{2.5.5}$$

$$p^\mu p^\nu P^{(2)}_{\mu\nu,\alpha\beta} = p^\mu p^\nu \left[\frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}\right] = 0, \tag{2.5.6}$$

$$p^\mu p^\nu P^{(0-\omega)}_{\mu\nu,\alpha\beta} = p^\mu p^\nu \omega_{\mu\nu}\omega_{\alpha\beta} = p^2\omega_{\alpha\beta}, \tag{2.5.7}$$

$$p^\mu p^\nu P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} = \frac{1}{\sqrt{3}}p^\mu p^\nu \left(\theta_{\mu\nu}\omega_{\alpha\beta} + \theta_{\alpha\beta}\omega_{\mu\nu}\right) = \frac{1}{\sqrt{3}}p^2\theta_{\alpha\beta}. \tag{2.5.8}$$

Com esses resultados, a eq. (2.5.3) torna-se:

$$p^2\left(a_4\omega_{\alpha\beta} + \frac{a_5}{\sqrt{3}}\theta_{\alpha\beta}\right)h^{\alpha\beta} = 0, \tag{2.5.9}$$

onde

$$\theta_{\alpha\beta}h^{\alpha\beta} = h - \omega_{\alpha\beta}h^{\alpha\beta}, \tag{2.5.10}$$

resultando em

$$a_4\omega_{\alpha\beta}h^{\alpha\beta} + \frac{a_5}{\sqrt{3}}\left(h - \omega_{\alpha\beta}h^{\alpha\beta}\right) = 0, \tag{2.5.11}$$

$$\left(a_4\sqrt{3} - a_5\right)\omega_{\alpha\beta}h^{\alpha\beta} + a_5 h = 0. \tag{2.5.12}$$

Lembrando que para a lagrangiana de Einstein-Hilbert, vale:

$$a_4 = \frac{1}{4\alpha}p^2, \qquad a_5 = -\frac{\sqrt{3}}{4\alpha}p^2, \tag{2.5.13}$$

encontramos:

$$\frac{2\sqrt{3}}{4\alpha}p^2\omega_{\alpha\beta}h^{\alpha\beta} - \frac{\sqrt{3}}{4\alpha}p^2 h = 0, \tag{2.5.14}$$

$$p_\alpha p_\beta h^{\alpha\beta} = \frac{1}{2}p^2 h, \tag{2.5.15}$$

que constitui uma primeira equação de vínculo sobre os graus de liberdade da teoria. Seguindo a investigação, realizamos uma segunda contração entre a eq. (2.5.1) e $\eta^{\mu\nu}$:

$$\eta^{\mu\nu}\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta}=0, \tag{2.5.16}$$

$$\eta^{\mu\nu}\left[a_1 P^{(1)}_{\mu\nu,\alpha\beta}+a_2 P^{(2)}_{\mu\nu,\alpha\beta}+a_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta}+a_4 P^{(0-\omega)}_{\mu\nu,\alpha\beta}+a_5 P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right]h^{\alpha\beta}=0. \tag{2.5.17}$$

Realizando a contração de $\eta^{\mu\nu}$ com os projetores, temos

$$\eta^{\mu\nu}P^{(0-\theta)}_{\mu\nu,\alpha\beta}=\frac{1}{3}\eta^{\mu\nu}\theta_{\mu\nu}\theta_{\alpha\beta}=\theta_{\alpha\beta}, \tag{2.5.18}$$

$$\eta^{\mu\nu}P^{(1)}_{\mu\nu,\alpha\beta}=\frac{1}{2}\eta^{\mu\nu}\left(\theta_{\mu\alpha}\omega_{\nu\beta}+\theta_{\mu\beta}\omega_{\nu\alpha}+\theta_{\nu\alpha}\omega_{\mu\beta}+\theta_{\nu\beta}\omega_{\mu\alpha}\right)=0, \tag{2.5.19}$$

$$\eta^{\mu\nu}P^{(2)}_{\mu\nu,\alpha\beta}=\left[\frac{1}{2}\left(\theta^{\nu}{}_{\alpha}\theta_{\nu\beta}+\theta^{\nu}{}_{\beta}\theta_{\nu\alpha}\right)-\theta_{\alpha\beta}\right]=\theta_{\alpha\beta}-\theta_{\alpha\beta}=0, \tag{2.5.20}$$

$$\eta^{\mu\nu}P^{(0-\omega)}_{\mu\nu,\alpha\beta}=\eta^{\mu\nu}\omega_{\mu\nu}\omega_{\alpha\beta}=\omega_{\alpha\beta}, \tag{2.5.21}$$

$$\eta^{\mu\nu}P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}=\frac{1}{\sqrt{3}}\eta^{\mu\nu}\left(\theta_{\mu\nu}\omega_{\alpha\beta}+\theta_{\alpha\beta}\omega_{\mu\nu}\right)=\frac{1}{\sqrt{3}}\left(3\omega_{\alpha\beta}+\theta_{\alpha\beta}\right). \tag{2.5.22}$$

Substituindo estes termos em (2.5.17), resulta

$$\left[\left(a_3+\frac{1}{\sqrt{3}}a_5\right)\theta_{\alpha\beta}+\left(a_4+\sqrt{3}a_5\right)\omega_{\alpha\beta}\right]h^{\alpha\beta}=0. \tag{2.5.23}$$

Sabemos que

$$a_3=-\left(\frac{4\alpha-3}{4\alpha}\right)p^2,\qquad a_4=\frac{1}{4\alpha}p^2 \text{ e }\qquad a_5=-\frac{\sqrt{3}}{4\alpha}p^2, \tag{2.5.24}$$

logo,

$$\left[\left(2\alpha-1\right)\theta_{\alpha\beta}+\omega_{\alpha\beta}\right]h^{\alpha\beta}=0, \tag{2.5.25}$$

$$p_\alpha p_\beta h^{\alpha\beta}=\frac{(2\alpha-1)}{2\left(\alpha-1\right)}p^2 h. \tag{2.5.26}$$

Combinando (2.5.26) com a eq. (2.5.15), decorre

$$h=0, \tag{2.5.27}$$

$$p_\alpha p_\beta h^{\alpha\beta}=0. \tag{2.5.28}$$

As eqs. (2.5.27) e (2.5.28) representam dois vínculos sobre as componentes de $h^{\mu\nu}$. Por último, calcularemos a contração,

$$p^{\mu}\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta}=0, \tag{2.5.29}$$

$$p^{\mu}\left[a_1 P^{(1)}_{\mu\nu,\alpha\beta}+a_2 P^{(2)}_{\mu\nu,\alpha\beta}+a_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta}+a_4 P^{(0-\omega)}_{\mu\nu,\alpha\beta}+a_5 P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right]h^{\alpha\beta}=0. \tag{2.5.30}$$

Novamente, fazendo a contração em cada uma das parcelas separadamente,

$$p^{\mu}P^{(0-\theta)}_{\mu\nu,\alpha\beta}=\frac{1}{3}p^{\mu}\theta_{\mu\nu}\theta_{\alpha\beta}=0, \tag{2.5.31}$$

$$p^{\mu}P^{(1)}_{\mu\nu,\alpha\beta}=\frac{1}{2}p^{\mu}\left(\theta_{\mu\alpha}\omega_{\nu\beta}+\theta_{\mu\beta}\omega_{\nu\alpha}+\theta_{\nu\alpha}\omega_{\mu\beta}+\theta_{\nu\beta}\omega_{\mu\alpha}\right)=\frac{1}{2}\left(\theta_{\nu\alpha}p_\beta+\theta_{\nu\beta}p_\alpha\right), \tag{2.5.32}$$

$$p^\mu P^{(2)}_{\mu\nu,\alpha\beta} = p^\mu \left[\frac{1}{2}\left(\theta_{\mu\alpha}\theta_{\nu\beta} + \theta_{\mu\beta}\theta_{\nu\alpha}\right) - \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}\right] = 0, \tag{2.5.33}$$

$$p^\mu P^{(0-\omega)}_{\mu\nu,\alpha\beta} = p^\mu \omega_{\mu\nu}\omega_{\alpha\beta} = p_\nu \omega_{\alpha\beta}, \tag{2.5.34}$$

$$p^\mu P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} = \frac{1}{\sqrt{3}} p^\mu \left(\theta_{\mu\nu}\omega_{\alpha\beta} + \theta_{\alpha\beta}\omega_{\mu\nu}\right) = \frac{1}{\sqrt{3}} p_\nu \theta_{\alpha\beta}, \tag{2.5.35}$$

que ao subsituir em (2.5.30), resulta

$$\left[\frac{1}{2}a_1\left(\theta_{\nu\alpha}p_\beta + \theta_{\nu\beta}p_\alpha\right) + a_4 p_\nu \omega_{\alpha\beta} + \frac{1}{\sqrt{3}} a_5 p_\nu \theta_{\alpha\beta}\right] h^{\alpha\beta} = 0, \tag{2.5.36}$$

onde

$$a_1 = \frac{1}{2\alpha}p^2, \qquad a_4 = \frac{1}{4\alpha}p^2, \qquad a_5 = -\frac{\sqrt{3}}{4\alpha}p^2. \tag{2.5.37}$$

Finalmente, encontramos

$$\begin{aligned} \left(2\theta_{\nu\beta}p_\alpha + p_\nu\omega_{\alpha\beta} - p_\nu\theta_{\alpha\beta}\right) h^{\alpha\beta} &= 0, \\ 2p_\alpha h^\alpha{}_\nu - p_\nu h &= 0, \end{aligned} \tag{2.5.38}$$

e, como já conhecemos $h = 0$, temos

$$p_\alpha h^\alpha{}_\nu = 0. \tag{2.5.39}$$

Esta equação estabelece um total de 4 vínculos, que ao juntarmos com aqueles expressos nas eqs. (2.5.27) e (2.5.28) totalizam 6 restrições. Assim, das 10 componentes de $h^{\mu\nu}$, restam apenas: $10 - 6 = 4$. Note que essa abordagem não elimina completamente os graus de liberdade não-físicos na teoria de Einstein-Hilbert.

2.6 Graus de liberdade (abordagem 2)

Vimos na seção anterior que o método de contagem adotado por [22], não funciona em teorias invariantes de calibre. Para obter corretamente as componentes propagantes de $h^{\mu\nu}$, seguiremos a mesma abordagem do Weinberg [18], com alguns elementos do Schutz [19]. Com o intuito de auferir corretamente os graus de liberdade da teoria gravitacional de Einstein-Hilbert, vamos considerar a solução para ondas planas (solução homogênea). As equações de Einstein, em sua forma homogênea ($T_{\mu\nu} = 0$), são dadas por:

$$R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0, \tag{2.6.1}$$

que em primeira ordem em $h^{\alpha\beta}$ pode ser escrita como (veja o anexo C):

$$\Box h_{\mu\nu} - \partial_\nu\partial_\beta h^\beta{}_\mu - \partial_\mu\partial_\beta h^\beta{}_\nu + \partial_\mu\partial_\nu h + \eta_{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \eta_{\mu\nu}\Box h = 0. \tag{2.6.2}$$

A eq. (2.6.2) pode ser reescrita usando a transformação,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}h, \tag{2.6.3}$$

que implica em

$$\Box \bar{h}_{\mu\nu} - \partial_\nu\partial_\beta \bar{h}^\beta{}_\mu - \partial_\mu\partial_\beta \bar{h}^\beta{}_\nu + \eta_{\mu\nu}\partial_\alpha\partial_\beta \bar{h}^{\alpha\beta} = 0. \tag{2.6.4}$$

A transformação dada por (2.6.3) é conhecida como traço-reverso, pois,

$$\bar{h}^{\mu}{}_{\mu} = \bar{h} = -h. \tag{2.6.5}$$

Como já mencionado na seção (2.3), a lagrangiana de Einstein-Hilbert é invariante sob a transformação,

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) - \partial_\nu \xi_\mu(x) - \partial_\mu \xi_\nu(x), \tag{2.6.6}$$

portanto, (2.6.2) também o será. Aplicando (2.6.6) em (2.6.3), podemos obter uma transformação que torna a eq. (2.6.4) invariante, dada a seguir

$$\bar{h}_{\mu\nu} \to \bar{h}_{\mu\nu} - \partial_\nu \xi_\mu - \partial_\mu \xi_\nu + \eta_{\mu\nu}\partial^\alpha \xi_\alpha, \tag{2.6.7}$$

cuja derivada fornece:

$$\partial^\mu \bar{h}_{\mu\nu} \to \partial^\mu \bar{h}_{\mu\nu} - \Box \xi_\nu. \tag{2.6.8}$$

Visto que a eq. (2.6.4) é invariante sob a transformação (2.6.7), há uma liberdade na escolha do vetor ξ_μ, vamos escolher um tal que a eq. (2.6.4) seja uma equação de onda. Escolhemos então,

$$\Box \xi_\nu = \partial^\mu \bar{h}_{\mu\nu} = 0, \tag{2.6.9}$$

que ao substituirmos na eq. (2.6.4), fornece a seguinte equação de onda:

$$\Box \bar{h}_{\mu\nu} = 0. \tag{2.6.10}$$

Vamos agora analisar as soluções da eq. (2.6.10). Sabemos que uma equação homogênea forma uma base que fornece as soluções para sua versão não-homogênea. Escrevemos a solução da eq. (2.6.10) na forma:

$$\bar{h}_{\mu\nu} = e_{\mu\nu}\exp(ip_\alpha x^\alpha) + e^*_{\mu\nu}\exp(-ip_\alpha x^\alpha), \tag{2.6.11}$$

onde $e_{\mu\nu}$ é um tensor simétrico conhecido como tensor de polarização. Aplicando essa solução na eq. (2.6.10), temos

$$p_\alpha p^\alpha = 0, \tag{2.6.12}$$

que reproduz o resultado (2.3.43). Já a condição (2.6.9) implica em

$$p_\mu e^{\mu}{}_{\nu} = \frac{1}{2} p_\nu e^{\mu}{}_{\nu}. \tag{2.6.13}$$

Como vimos, $e_{\mu\nu}$ é um tensor simétrico, logo possui apenas dez componentes independentes. Tal simetria decorre diretamente do fato de $\bar{h}_{\mu\nu}$ também ser um tensor simétrico. Assim, tendo em vista a eq. (2.6.13), as componentes independentes de $e_{\mu\nu}$ se reduzem a apenas seis. No entanto, há apenas dois graus de liberdade propagantes dentro da teoria gravitacional, ou seja, somente duas dessas componentes têm significado físico. Sabemos que a equação (2.6.10), dentro da escolha denotada pela eq. (2.6.9), é invariante sob a transformação (2.6.7), vamos propor uma forma específica para o vetor ξ_μ, e verificar como essa mudança afeta a solução (2.6.11). Seja,

$$\xi'_\mu = i\xi_\mu \exp(ip_\alpha x^\alpha) - i\xi^*_\mu \exp(-ip_\alpha x^\alpha), \tag{2.6.14}$$

temos:

$$\partial_\nu \xi'_\mu = -ip_\nu \xi_\mu \exp(ip_\alpha x^\alpha) - ip_\mu \xi^*_\mu \exp(-ip_\alpha x^\alpha). \tag{2.6.15}$$

Logo,

$$\bar{h}'_{\mu\nu}(x) = \bar{h}_{\mu\nu}(x) - \partial_\nu \xi'_\mu(x) - \partial_\mu \xi'_\nu(x) \tag{2.6.16}$$

torna-se:

$$\bar{h}'_{\mu\nu}(x) = e_{\mu\nu} \exp(ip_\alpha x^\alpha) + e^*_{\mu\nu} \exp(-ip_\alpha x^\alpha) - \partial_\nu \xi'_\mu(x) - \partial_\mu \xi'_\nu(x) \,. \tag{2.6.17}$$

Ao substituir (2.6.15) na expressão acima, obtemos o resultado

$$\bar{h}'_{\mu\nu}(x) = e'_{\mu\nu} \exp(ip_\alpha x^\alpha) + e'^*_{\mu\nu} \exp(-ip_\alpha x^\alpha) \,, \tag{2.6.18}$$

onde

$$e'_{\mu\nu} = e_{\mu\nu} + p_\nu \xi_\mu + p_\mu \xi_\nu . \tag{2.6.19}$$

Observe que a solução (2.6.18) possui a mesma forma de (2.6.11). Portanto, $e'_{\mu\nu}$ e $e_{\mu\nu}$ representam a mesma física independentemente da escolha do vetor ξ_μ. Além disso, uma escolha arbitrária de ξ_μ possui quatro componentes distintas (uma para cada índice livre). Logo, as componentes independentes de $h_{\mu\nu}$ se resumem a apenas duas $(6-4=2)$, que são os dois graus de liberdade físicos na teoria de Einstein-Hilbert.

CAPÍTULO 3

Alguns modelos alternativos de gravitação

A teoria da relatividade geral tem sido amplamente testada por diversos experimentos ao longo dos anos, e sempre tem se provado bem-sucedida não apenas na que diz respeito à verificação experimental, mas também na previsão de novos efeitos como as ondas gravitacionais. No entanto, a despeito de seu sucesso experimental, a TRG proposta por Einstein não se enquadra no rol das teorias de campo renormalizáveis, sendo incompatível com o Modelo Padrão. Neste capítulo abordaremos o cálculo do propagador em duas teorias gravitacionais alternativas à de Einstein-Hilbert: gravitação com termos quadráticos e em (2+1) dimensões.

Uma ação que contenha termos quadráticos em teorias gravitacionais, pode ser genericamente escrita como [10]:

$$S = \int d^n x \sqrt{-g} \left(R + \alpha_1 R^2 + \alpha_2 R^{\mu\nu} R_{\mu\nu} + \alpha_3 g^{\mu\nu} \partial_\mu R \partial_\nu R + ... \right),$$

onde os coeficientes α_i representam constantes de acoplamento e a reticências termos diversos que podem ser construídos pela combinação dos escalar e tensor de Ricci. Neste trabalho abordaremos apenas as três primeiras parcelas, o restante será desconsiderado por não possuírem contribuição significativa, mas apenas complicações indesejadas.

3.1 Propagador para teorias gravitacionais quadráticas

Dando continuidade à aplicação do método geral para a obtenção do propagador de teorias gravitacionais, exemplificaremos seu cálculo para um caso diferente do anterior: uma teoria gravitacional com termos quadráticos em R e $R_{\mu\nu}$ na ação. Tais teorias surgiram da tentativa de encontrar uma alternativa viável dotada de altas derivadas [13] e [14], teoricamente consistente com os dados experimentais e que seja renormalizável. Tais teorias, por sua vez, trazem consigo um outro fator: modos massivos que surgem por conta dos termos de altas derivadas associados com os termos R^2 e $R^{\mu\nu}R_{\mu\nu}$. Tais modos podem ser um problema, pois a massa do gráviton é extremamente limitada. No entanto, a despeito dessa problemática, calcularemos o propagador neste cenário, onde notaremos o surgimento dos modos massivos que decorrem diretamente dos polos obtidos no propagador.

A lagrangiana para este caso é a seguinte [14] e [17]

$$\mathcal{L} = \mathcal{L}_{E-H} + \frac{\sigma}{2} R^2 + \frac{\gamma}{2} R^{\mu\nu} R_{\mu\nu} + \mathcal{L}_{G-F}. \tag{3.1.1}$$

As constantes σ e γ possuem dimensão de massa igual a -2 ($[\sigma] = [\gamma] = -2$). Já conhecemos a forma bilinear da parte correspondente à ação Einstein-Hilbert e à fixação de calibre. Resta então lidar com os termos quadráticos dos escalar e tensor de Ricci. Usando a forma linearizada do escalar de Ricci, escrevemos:

$$\begin{aligned} R^2 &= (\partial_\mu\partial_\nu h^{\mu\nu} - \Box h)\left(\partial_\alpha\partial_\beta h^{\alpha\beta} - \Box h\right), \\ &= \left(\partial_\mu\partial_\nu h^{\mu\nu}\partial_\alpha\partial_\beta h^{\alpha\beta} - \partial_\mu\partial_\nu h^{\mu\nu}\Box h\right) - \Box h\partial_\alpha\partial_\beta h^{\alpha\beta} + \Box h\Box h. \end{aligned} \tag{3.1.2}$$

Seguindo os mesmos passos já apresentado em seções anteriores, temos

$$R^2 = h^{\mu\nu}\left(-\eta_{\mu\nu}\partial_\alpha\partial_\beta\Box - \eta_{\alpha\beta}\partial_\mu\partial_\nu\Box + \eta_{\mu\nu}\eta_{\alpha\beta}\Box^2 + \partial_\mu\partial_\nu\partial_\alpha\partial_\beta\right)h^{\alpha\beta}. \tag{3.1.3}$$

Partindo da forma linearizada do tensor de Ricci consideramos a seguinte expressão para o quadrado do tensor de Ricci:

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} = \frac{1}{2}&\left(\partial_\beta\partial_\nu h^\beta{}_\mu - \Box h_{\mu\nu} - \partial_\mu\partial_\nu h + \partial_\beta\partial_\mu h^\beta{}_\mu\right) \\ &\times \frac{1}{2}\left(\partial^\alpha\partial^\nu h^\mu{}_\alpha - \Box h^{\mu\nu} - \partial^\mu\partial^\nu h + \partial^\alpha\partial^\mu h^\nu{}_\alpha\right). \end{aligned} \tag{3.1.4}$$

Desenvolvendo e renomeando os índices contraídos de forma conveniente, resulta

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} = \frac{1}{4}&\left(2\partial_\beta\partial_\nu h^\beta{}_\mu\partial^\alpha\partial^\nu h^\mu{}_\alpha + 2\partial_\beta\partial_\mu h^\beta{}_\mu\partial^\alpha\partial^\mu h^\nu{}_\alpha\right. \\ &- 4\partial_\mu\partial_\nu h\partial^\alpha\partial^\mu h^\nu{}_\alpha - 4\Box h_{\mu\nu}\partial^\alpha\partial^\mu h^\nu{}_\alpha \\ &\left.+2\Box h_{\mu\nu}\partial^\mu\partial^\nu h + \Box h_{\mu\nu}\Box h^{\mu\nu} + \partial_\mu\partial_\nu h\partial^\mu\partial^\nu h\right). \end{aligned} \tag{3.1.5}$$

Novamente, podemos reescrever estes termos por meio do recurso da derivada total,

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} = \frac{1}{4}&\left(-2h_{\mu\beta}\Box\partial^\alpha\partial^\mu h^\beta{}_\alpha - 2h\Box\partial_\mu\partial_\nu h^{\mu\nu}\right. \\ &\left.+2h^{\mu\nu}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta h^{\alpha\beta} + h^{\mu\nu}\Box^2 h_{\mu\nu} + h\Box^2 h\right). \end{aligned} \tag{3.1.6}$$

Usamos a métrica para subir ou descer convenientemente índices tensoriais,

$$\begin{aligned} R^{\mu\nu}R_{\mu\nu} = h^{\mu\nu}&\left(-\frac{1}{2}\eta_{\nu\beta}\partial_\mu\partial_\alpha\Box - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial_\beta\Box\right. \\ &\left.+\frac{1}{2}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta + \frac{1}{4}\eta_{\mu\alpha}\eta_{\nu\beta}\Box^2 + \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box^2\right)h^{\alpha\beta}, \end{aligned} \tag{3.1.7}$$

de modo que podemos reescrever a lagrangiana total na forma

$$\mathcal{L} = \frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta}, \tag{3.1.8}$$

onde o operador $\mathcal{O}_{\mu\nu,\alpha\beta}$ é dado por

$$\begin{aligned} \mathcal{O}_{\mu\nu,\alpha\beta} = &\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \eta_{\mu\nu}\partial_\alpha\partial_\beta - \frac{1}{2}\eta_{\mu\alpha}\eta_{\nu\beta}\Box + \eta_{\nu\alpha}\partial_\beta\partial_\mu \\ &+ \sigma\left(-\eta_{\mu\nu}\partial_\alpha\partial_\beta\Box - \eta_{\alpha\beta}\partial_\mu\partial_\nu\Box + \eta_{\mu\nu}\eta_{\alpha\beta}\Box^2 + \partial_\mu\partial_\nu\partial_\alpha\partial_\beta\right) \\ &+ \gamma\left(-\frac{1}{2}\eta_{\nu\beta}\partial_\mu\partial_\alpha\Box - \frac{1}{2}\eta_{\mu\nu}\partial_\alpha\partial_\beta\Box + \frac{1}{2}\partial_\mu\partial_\nu\partial_\alpha\partial_\beta + \frac{1}{4}\eta_{\mu\alpha}\eta_{\nu\beta}\Box^2 + \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box^2\right) \\ &+ \frac{1}{\alpha}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta - \eta_{\nu\beta}\partial_\mu\partial_\alpha - \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box\right). \end{aligned} \tag{3.1.9}$$

Novamente, surgem termos que necessitam ser simetrizados. Lançamos mão das das eqs. (2.3.24) - (2.3.26) para obter

$$
\begin{aligned}
\mathcal{O}_{\mu\nu,\alpha\beta} = {} & \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box - \frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right) - \frac{1}{4}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right)\Box \\
& + \frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right) \\
& + \sigma\left[-\frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right)\Box - \frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right)\Box + \eta_{\mu\nu}\eta_{\alpha\beta}\Box^2 + \partial_\mu\partial_\nu\partial_\alpha\partial_\beta\right] \\
& + \frac{\gamma}{2}\left[-\frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right)\Box\right. \\
& - \frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right)\Box + \partial_\mu\partial_\nu\partial_\alpha\partial_\beta + \frac{1}{4}\left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\mu\beta}\eta_{\nu\alpha}\right)\Box^2 \\
& \left.+\frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\Box^2\right] + \frac{1}{\alpha}\left[\frac{1}{2}\left(\eta_{\mu\nu}\partial_\alpha\partial_\beta + \eta_{\alpha\beta}\partial_\mu\partial_\nu\right)\right. \\
& \left.-\frac{1}{4}\left(\eta_{\nu\alpha}\partial_\mu\partial_\beta + \eta_{\mu\alpha}\partial_\nu\partial_\beta + \eta_{\nu\beta}\partial_\mu\partial_\alpha + \eta_{\mu\beta}\partial_\nu\partial_\alpha\right) - \frac{1}{4}\eta_{\mu\nu}\eta_{\alpha\beta}\Box\right]. \qquad (3.1.10)
\end{aligned}
$$

O operador (3.1.10) pode ser expresso em termos dos projetores de Barnes-Rivers (2.2.8-2.2.12) usando-se relações do tipo (2.3.31-2.3.34). Encontramos:

$$
\begin{aligned}
\mathcal{O}_{\mu\nu,\alpha\beta} = {} & \frac{1}{2}\left(3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box - \frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box \\
& - \frac{1}{2}\left(P^{(0-\theta)} + P^{(0-\omega)} + P^{(1)} + P^{(2)}\right)\Box + \frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right)\Box \\
& + \sigma\left[-\frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box^2 - \frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box^2\right. \\
& \left.+\left(3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box^2 + \Box^2 P^{(0-\omega)}\right] \\
& \frac{\gamma}{2}\left[-\frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right)\Box^2 - \frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\right. \\
& + \Box^2 P^{(0-\omega)} + \frac{1}{2}\left(P^{(0-\theta)} + P^{(0-\omega)} + P^{(1)} + P^{(2)}\right)\Box^2 \\
& \left.+\frac{1}{2}\left(3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\Box^2\right] \\
& + \frac{1}{\alpha}\left[\frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right) - \frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right)\right. \\
& \left.-\frac{1}{4}\left(3P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right)\right]\Box. \qquad (3.1.11)
\end{aligned}
$$

Agrupando os termos semelhantes e fazendo as simplificações cabíveis, resulta

$$
\begin{aligned}
\mathcal{O}_{\mu\nu,\alpha\beta} = {} & \left[\frac{(4\alpha-3)\Box + 4\alpha\left(3\sigma+\gamma\right)\Box^2}{4\alpha}\right]P^{(0-\theta)} - \frac{\Box}{4\alpha}P^{(0-\omega)} \\
& - \frac{\Box}{2\alpha}P^{(1)} - \left(\frac{2\Box - \gamma\Box^2}{4}\right)P^{(2)} - \frac{\sqrt{3}\Box}{4\alpha}P^{(0-\theta\omega)}. \qquad (3.1.12)
\end{aligned}
$$

Assim, o propagador para a teoria gravitacional quadrática dada na lagrangiana (3.1.1)

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = -\frac{2\alpha}{\Box}P^{(1)} - \frac{4}{2\Box - \gamma\Box^2}P^{(2)} - \frac{4\alpha}{(3\sigma+\gamma)\Box^3 + \Box^2}\left[-\frac{\Box}{4\alpha}P^{(0-\theta)} + \frac{(4\alpha-3)\Box + 4\alpha(3\sigma+\gamma)\Box^2}{4\alpha}P^{(0-\omega)} - \frac{\sqrt{3}\Box}{4\alpha}P^{(0-\theta\omega)}\right], \quad (3.1.13)$$

que pode ser reescrito como

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = -\frac{2\alpha}{\Box}P^{(1)} - \frac{4}{2\Box - \gamma\Box^2}P^{(2)} - \frac{4\alpha}{\Box}P^{(0-\omega)} + \frac{1}{(3\sigma+\gamma)\Box^2 + \Box}\left(P^{(0-\theta)} + 3P^{(0-\omega)} + \sqrt{3}P^{(0-\theta\omega)}\right). \quad (3.1.14)$$

Escrevemos também o propagador na forma de Feynman no espaço dos momentos

$$\Delta_{\mu\nu,\alpha\beta} = i\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}, \quad (3.1.15)$$

$$\Delta_{\mu\nu,\alpha\beta} = \frac{i}{p^2}\left[2\alpha P^{(1)}_{\mu\nu,\alpha\beta} + \frac{4}{2+\gamma p^2}P^{(2)}_{\mu\nu,\alpha\beta} + 4\alpha P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \frac{1}{(3\sigma+\gamma)p^2 - 1}\left(P^{(0-\theta)}_{\mu\nu,\alpha\beta} + 3P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \sqrt{3}P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right)\right], \quad (3.1.16)$$

cujos polos são

$$p^2 = 0, \quad (3.1.17)$$
$$2 + \gamma p^2 = 0, \quad (3.1.18)$$
$$(3\sigma+\gamma)p^2 - 1 = 0. \quad (3.1.19)$$

Como havíamos mencionado no início desta seção, os novos polos que emergem desta estrutura estão associados aos termos de altas derivadas. O primeiro polo que surge fornece a seguinte relação de dispersão

$$p^2 = -m_1^2,$$
$$p_0^2 - |\mathbf{p}|^2 = -m_1^2,$$
$$p_0 = \pm\sqrt{|\mathbf{p}|^2 - m_1^2}, \quad (3.1.20)$$

onde $m_1^2 = \frac{2}{\gamma}$. A velocidade de grupo associada vale

$$u_g = \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 - m_1^2}}. \quad (3.1.21)$$

Vemos que $u_g > 1$ para $\gamma > 0$, o que implica em violação da causalidade. Podemos ainda calcular a velocidade de frente de onda (u_f)

$$u_f = \lim_{|\mathbf{p}|\to\infty}\frac{p_0}{|\mathbf{p}|} = \sqrt{1 - \frac{m_1^2}{|\mathbf{p}|^2}} = 1. \quad (3.1.22)$$

Ambas as condições $u_g \leq 1$ e $u_f \leq 1$ são necessárias para assegurar a a causalidade do modo propagante. Logo, para esse modo massivo ser compatível com os requisitos da causalidade, deve valer $\gamma < 0$.

Para o segundo polo, temos a seguinte relação de dispersão

$$\begin{aligned}
p^2 &= m_2^2, \\
p_0^2 &= |\mathbf{p}|^2 + m_2^2, \\
p_0 &= \pm\sqrt{|\mathbf{p}|^2 + m_2^2}
\end{aligned} \tag{3.1.23}$$

onde $m_2^2 = \frac{1}{(3\sigma+\gamma)}$, que fornece velocidade de grupo subluminal:

$$u_g = \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + m_2^2}} < 1. \tag{3.1.24}$$

Neste caso, a velocidade de grupo é subluminal, sempre que $(3\sigma + \gamma) > 0$. Em resumo, para assegurar que os modos massivos dessa teoria sejam causais, deve valer:

$$\gamma < 0, \tag{3.1.25}$$

$$3\sigma + \gamma > 0. \tag{3.1.26}$$

3.2 Unitariedade tree-level

Para analisar a consistência deste modelo, faremos a análise por meio do mecanismo de saturação do propagador de Feynman com correntes externas. Isso nos permite encontrar uma expressão que forneça a norma das excitações associadas a cada uma das relações de dispersão, a unitariedade é assegurada quando a norma das excitações de todos os estados é positiva. Podemos identificar a norma dessas excitações observando o sinal da parte imaginária da saturação do propagador dada por

$$SP = J^{\mu\nu} Res\left[\Delta_{\mu\nu,\alpha\beta}\right] J^{\alpha\beta}, \tag{3.2.1}$$

onde $J^{\mu\nu}$ é um tensor simétrico que descreve uma corrente externa conservada $p_\mu J^{\mu\nu} = 0$.

Uma vez observado que a corrente externa é conservada, nem todos os projetores terão saturação não-nula, isso só ocorrerá nos termos não proporcionais a $\omega_{\mu\nu}$. Assim, como já obtido, teremos

$$J^{\mu\nu} P^{(2)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} = J_{\alpha\beta} J^{\alpha\beta} - \frac{1}{3}\left(J^{\alpha}{}_{\alpha}\right)^2, \tag{3.2.2}$$

$$J^{\mu\nu} P^{(0-\theta)}_{\mu\nu,\alpha\beta} J^{\alpha\beta} = \frac{1}{3}\left(J^{\alpha}{}_{\alpha}\right)^2. \tag{3.2.3}$$

Usando o propagador de Feynman (3.1.16), podemos escrever o propagador saturado por correntes conservadas como sendo

$$SP = iRes\left[\frac{1}{p^2}\left(2m_1^2 \frac{J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{3}\left(J^{\alpha}{}_{\alpha}\right)^2}{p^2 + m_1^2} + \frac{m_2^2}{3}\frac{\left(J^{\alpha}{}_{\alpha}\right)^2}{p^2 - m_2^2}\right)\right]. \tag{3.2.4}$$

Devemos então avaliar o resíduo da saturação nos dois termos e em cada um dos polos de interesse. Assim, o resíduo para o polo $p^2 = 0$ é

$$\begin{aligned}
Res|_{p^2=0} &= 2\left(J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{3}\left(J^{\alpha}{}_{\alpha}\right)^2\right) - \frac{1}{3}\left(J^{\alpha}{}_{\alpha}\right)^2, \\
&= 2J_{\alpha\beta}J^{\alpha\beta} - \left(J^{\alpha}{}_{\alpha}\right)^2.
\end{aligned}$$

Retomando o resultado da eq. (2.4.11) , temos

$$
\begin{aligned}
Res|_{p^2=0} &= \frac{2m_1^2}{m_1^2}\left[\frac{2}{3}\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2}+\frac{1}{2}J_{aa}\right)^2 - 2\frac{(p_c J_{ca})^2}{|\mathbf{p}|} + (J_{ca})^2 - \frac{1}{2}(J_{aa})^2\right], \\
&= \left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2}+J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{|\mathbf{p}|} - 2(J_{aa})^2, \qquad (3.2.5)
\end{aligned}
$$

que é idêntico ao resultado obtido para a saturação dentro da teoria de Einstein-Hilbert. Podemos então concluir que este modo é unitário. Avaliamos o segundo polo $p^2 + m_1^2 = 0$,

$$
Res|_{p^2+m_1^2=0} = -2\left(J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{3}(J^{\alpha}{}_{\alpha})^2\right). \qquad (3.2.6)
$$

Somamos e subtraímos o termo $\frac{1}{3}(J^{\alpha}{}_{\alpha})^2$, a fim de obter:

$$
Res|_{p^2+m_1^2=0} = -2\left(J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{2}(J^{\alpha}{}_{\alpha})^2\right) - \frac{1}{3}(J^{\alpha}{}_{\alpha})^2,
$$

que implica em

$$
Res|_{p^2+m_1^2=0} = -\left[\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2 - m_1^2}+J_{aa}\right)^2 + 2(J_{ca})^2 - 4\frac{(p_c J_{ca})^2}{\sqrt{|\mathbf{p}|^2 - m_1^2}} - 2(J_{aa})^2\right] - \frac{1}{3}\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2 - m_1^2} - J_{aa}\right)^2, \qquad (3.2.7)
$$

onde usamos

$$
(J^{\alpha}{}_{\alpha})^2 = \left(\frac{p_a p_c J_{ca}}{p_0^2} - J_{aa}\right)^2.
$$

Observe que neste caso conseguimos escrever o resíduo do propagador em termos de duas parcelas positivas, precedidas de sinais negativos indicando que esse modo é não unitário. Finalmente, para $p^2 - m_2^2 = 0$:

$$
\begin{aligned}
Res|_{p^2-m_2^2=0} &= \frac{1}{3}(J^{\alpha}{}_{\alpha})^2, \qquad (3.2.8) \\
Res|_{p^2-m_2^2=0} &= \frac{1}{3}\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2 + m_2^2} - J_{aa}\right)^2, \qquad (3.2.9)
\end{aligned}
$$

que é um termo claramente positivo, implicando na unitariedade do modo. Como pode ser observado nos resultados acima, alguns modos são provêm excitações com norma negativa incorrendo na conclusão de a teoria de gravitação quadrática ser não-unitária, tal afirmação está de acordo com [17] .

3.3 Propagador para gravitação de Chern-Simons

A gravitação de Chern-Simons foi introduzida inicialmente por Giddings *et. al* em 1984 [20], como uma preparação para o estudo de modelos de gravitação quântica. Neste trabalho, foram obtidas informações relevantes sobre a estrutura das equações de Einstein em diferentes situações, e a sua principal conclusão é a seguinte. Em três dimensões, a teoria da relatividade geral não possui graus de liberdade propagantes, ou seja, ao findarmos o cálculo do propagador encontraremos um polo que não estará associado a qualquer modo de propagação.

Temos uma problemática semelhante na teoria gravitacional linearizada: a solução das equações de campo usando a aproximação de campo fraco (weak-field) possui solução trivial, isso indica que não existem ondas gravitacionais em três dimensões.

Aparentemente não possuir graus de liberdade propagantes parece indicar um problema, mas é justamente esse fato que torna a gravitação de Chern-Simons interessante. Ao se modificar uma teoria, torna-se relativamente complicado separar os graus de liberdade dinâmicos próprios da teoria daqueles advindos de sua modificação, enquanto no cenário da gravitação tridimensional esse problema desaparece por não haver graus de liberdade.

O estudo de teorias gravitacionais em baixas dimensões pode ser realizado por meio do acoplamento do termo de gravidade de Chern-Simons [15] à lagrangiana de Einstein-Hilbert

$$\mathcal{L} = \mathcal{L}_{E-H} + \mathcal{L}_{C-S} + \mathcal{L}_{G-F}, \tag{3.3.1}$$

sendo $\mathcal{L}_{C-S}$ a lagrangiana de Chern-Simons:

$$\mathcal{L}_{C-S} = \frac{1}{2\tau}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\kappa\lambda}\left(\partial_{\mu}\Gamma^{\kappa}{}_{\rho\nu} + \frac{2}{3}\Gamma^{\kappa}{}_{\gamma\mu}\Gamma^{\gamma}{}_{\rho\nu}\right), \tag{3.3.2}$$

onde τ é uma constante com dimensão de massa 1.

3.3.1 Extensão da base de Barnes-Rivers

A inserção do termo de Chern-Simons torna a base de Barnes-Rivers insuficiente para o cálculo do propagador, ou seja, deixamos de ter uma álgebra fechada para os operadores de projeção de spin em (2+1) dimensões. Assim, surge a necessidade de estendê-la a fim de tornar possível o cálculo do propagador. Nesse intuito, vamos escrever a lagrangiana de Chern-Simons na sua forma bilinear e então extrair o novo operador que irá completar a base.

A forma linearizada da lagrangiana (3.3.2) só terá contribuições da primeira parcela, por ser é quadrática no campo $h^{\mu\nu}$, enquanto a terceira parcela é cúbica no símbolo de Christoffel e, consequentemente em $h^{\mu\nu}$. O termo de interesse é

$$\mathcal{L}_{C-S} = \frac{1}{2\tau}\epsilon^{\lambda\mu\nu}\Gamma^{\rho}{}_{\kappa\lambda}\partial_{\mu}\Gamma^{\kappa}{}_{\rho\nu}. \tag{3.3.3}$$

Usando a forma linearizada do Símbolo de Christoffel, temos

$$\mathcal{L}_{C-S} = \frac{1}{8\tau}\epsilon^{\lambda\mu\nu}\eta^{\rho\varpi}\eta^{\kappa\delta}\left(\partial_{\kappa}h_{\lambda\varpi} + \partial_{\lambda}h_{\kappa\varpi} - \partial_{\varpi}h_{\lambda\kappa}\right)\left(\partial_{\mu}\partial_{\rho}h_{\nu\delta} + \partial_{\mu}\partial_{\nu}h_{\rho\delta} - \partial_{\mu}\partial_{\delta}h_{\rho\nu}\right), \tag{3.3.4}$$

desenvolvendo e desprezando as parcelas com derivadas sucessivas em μ e ν, (que são nulas por causa do símbolo de Levi-Civita), encontramos:

$$\begin{aligned}\mathcal{L}_{C-S} = \frac{1}{8\tau}\epsilon^{\lambda\mu\nu}\Big[&\left(\partial^{\delta}h^{\rho}{}_{\lambda}\partial_{\mu}\partial_{\rho}h_{\nu\delta} - \partial^{\delta}h^{\rho}{}_{\lambda}\partial_{\mu}\partial_{\delta}h_{\rho\nu}\right) + \left(\partial_{\lambda}h^{\delta\rho}\partial_{\mu}\partial_{\rho}h_{\nu\delta} - \partial_{\lambda}h^{\delta\rho}\partial_{\mu}\partial_{\delta}h_{\rho\nu}\right)\\ &- \left(\partial^{\rho}h^{\delta}{}_{\lambda}\partial_{\mu}\partial_{\rho}h_{\nu\delta} - \partial^{\rho}h^{\delta}{}_{\lambda}\partial_{\mu}\partial_{\delta}h_{\rho\nu}\right)\Big].\end{aligned}$$

Observe que o segundo termo entre parênteses se anula por uma simples redefinição nos índices contraídos e que as parcelas restantes se somam pelo mesmo motivo, de modo que obtemos:

$$\mathcal{L}_{C-S} = \frac{1}{4\tau}\epsilon^{\lambda\mu\nu}\left(\partial^{\delta}h^{\rho}{}_{\lambda}\partial_{\mu}\partial_{\rho}h_{\nu\delta} - \partial^{\delta}h^{\rho}{}_{\lambda}\partial_{\mu}\partial_{\delta}h_{\rho\nu}\right). \tag{3.3.5}$$

Integramos por partes e usamos a métrica para extrair a forma bilinear

$$\mathcal{L}_{C-S} = \frac{1}{4\tau}\epsilon_{\lambda\mu\nu}h^{\rho\lambda}\partial^{\mu}\left(\Box\eta_{\rho\delta} - \partial_{\rho}\partial_{\delta}\right)h^{\delta\nu}, \tag{3.3.6}$$

que pode ser expressa em termos do projetor transversal:

$$\mathcal{L}_{C-S} = \frac{1}{4\tau}h^{\rho\lambda}\Box\partial^{\mu}\epsilon_{\lambda\mu\nu}\theta_{\rho\delta}h^{\delta\nu}. \tag{3.3.7}$$

Note que a expressão acima não está simetrizada, o processo de simetrização ocorre de acordo com os procedimentos já estudados, fornecendo:

$$\mathcal{L}_{C-S} = \frac{1}{4\tau}h^{\rho\lambda}\Box\partial^{\mu}\left(\frac{\epsilon_{\lambda\mu\nu}\theta_{\rho\delta} + \epsilon_{\delta\mu\nu}\theta_{\rho\lambda} + \epsilon_{\rho\mu\delta}\theta_{\nu\lambda} + \epsilon_{\rho\mu\lambda}\theta_{\nu\delta}}{4}\right)h^{\delta\nu}, \tag{3.3.8}$$

ou ainda

$$\mathcal{L}_{C-S} = \frac{1}{2}h^{\rho\lambda}\left(\frac{\Box}{4\tau}S_{\rho\lambda,\delta\nu}\right)h^{\delta\nu}, \tag{3.3.9}$$

onde

$$S_{\rho\lambda,\delta\nu} = \frac{1}{2}\left(S_{\lambda\nu}\theta_{\rho\delta} + S_{\delta\nu}\theta_{\rho\lambda} + S_{\rho\delta}\theta_{\nu\lambda} + S_{\rho\lambda}\theta_{\nu\delta}\right), \tag{3.3.10}$$

é o novo operador projeção de spin que irá compor a base de Barnes-Rivers. Neste operador, temos

$$S_{\alpha\beta} = \partial^{\gamma}\epsilon_{\alpha\gamma\beta}, \tag{3.3.11}$$

que é o mesmo termo de Chern-Simons do eletromagnetismo. Observe ainda que $S_{\rho\lambda,\delta\nu}$ possui as seguintes simetrias

$$S_{\rho\lambda,\delta\nu} = S_{\lambda\rho,\nu\delta} = -S_{\delta\nu,\rho\lambda}.$$

O sinal negativo na última permutação está relacionado com a própria estrutura antissimétrica do símbolo de Levi-Civita que compõe o termo de Chern-Simons.

De posse do operador adicional, devemos obter uma álgebra fechada de projetores que viabilize o cálculo do propagador dessa teoria definida em (2+1) dimensões. Para isso, vamos calcular as contrações não nulas entre $S_{\mu\nu,\alpha\beta}$ e a base de Barnes-Rivers, que em (2+1) dimensões torna-se

$$P^{(1)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}\left(\theta_{\mu\kappa}\omega_{\nu\lambda} + \theta_{\mu\lambda}\omega_{\nu\kappa} + \theta_{\nu\kappa}\omega_{\mu\lambda} + \theta_{\nu\lambda}\omega_{\mu\kappa}\right), \tag{3.3.12}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}\left(\theta_{\mu\kappa}\theta_{\nu\lambda} + \theta_{\mu\lambda}\theta_{\nu\kappa} - \theta_{\mu\nu}\theta_{\kappa\lambda}\right), \tag{3.3.13}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda} = \frac{1}{2}\theta_{\mu\nu}\theta_{\kappa\lambda}, \tag{3.3.14}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\nu}\omega_{\kappa\lambda}, \tag{3.3.15}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{2}}\left(\theta_{\mu\nu}\omega_{\kappa\lambda} + \theta_{\kappa\lambda}\omega_{\mu\nu}\right), \tag{3.3.16}$$

$$S_{\mu\nu,\kappa\lambda} = \frac{1}{2}\left(S_{\mu\kappa}\theta_{\nu\lambda} + S_{\mu\lambda}\theta_{\nu\kappa} + S_{\nu\kappa}\theta_{\mu\lambda} + S_{\nu\lambda}\theta_{\mu\kappa}\right). \tag{3.3.17}$$

Em (3.3.12) o fator 1/2 já era esperado, afinal o cenário tridimensional não afeta este projetor bem como (3.3.15). Já em $P^{(2)}_{\mu\nu,\kappa\lambda}$, $P^{(0-\theta)}_{\mu\nu,\kappa\lambda}$ e $P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}$ a história é diferente. Se olharmos para a definição dos operadores de Barnes-Rivers em D dimensões (2.2.1-2.2.5), podemos notar que a dimensão do espaço-tempo abordado altera a forma destes projetores.

Como já discutido, a obtenção do propagador requer o cálculo das contrações dos operadores que compõem a álgebra. Neste sentido, devemos calcular as contrações que envolvem $S_{\mu\nu,\kappa\lambda}$. Afirmamos que apenas duas serão não nulas, a primeira delas é:

$$
\begin{aligned}
S_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} &= \frac{\partial^\sigma}{2}\left(\epsilon_{\mu\sigma\kappa}\theta_{\nu\lambda}+\epsilon_{\mu\sigma\lambda}\theta_{\nu\kappa}+\epsilon_{\nu\sigma\kappa}\theta_{\mu\lambda}+\epsilon_{\nu\sigma\lambda}\theta_{\mu\kappa}\right)\times\frac{1}{2}\left(\theta^\kappa{}_\alpha\theta^\lambda{}_\beta+\theta^\lambda{}_\alpha\theta^\kappa{}_\beta-\theta_{\alpha\beta}\theta^{\kappa\lambda}\right), \\
&= \frac{\partial^\sigma}{4}\left(\epsilon_{\mu\sigma\kappa}\theta_{\nu\lambda}+\epsilon_{\mu\sigma\lambda}\theta_{\nu\kappa}+\epsilon_{\nu\sigma\kappa}\theta_{\mu\lambda}+\epsilon_{\nu\sigma\lambda}\theta_{\mu\kappa}\right)\left(\theta^\kappa{}_\alpha\theta^\lambda{}_\beta+\theta^\lambda{}_\alpha\theta^\kappa{}_\beta\right).
\end{aligned} \tag{3.3.18}
$$

Desenvolvendo o produto e agrupando os termos semelhantes, resulta

$$
S_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{\partial^\sigma}{2}\left(\epsilon_{\mu\sigma\kappa}\theta_{\nu\beta}\theta^\kappa{}_\alpha+\epsilon_{\mu\sigma\lambda}\theta_{\nu\alpha}\theta^\lambda{}_\beta+\epsilon_{\nu\sigma\kappa}\theta_{\mu\beta}\theta^\kappa{}_\alpha+\epsilon_{\nu\sigma\lambda}\theta_{\mu\alpha}\theta^\lambda{}_\beta\right), \tag{3.3.19}
$$

onde reescrevemos as parcelas de cada uma das parcelas de (3.3.19) usando

$$
\partial^\sigma\epsilon_{\mu\sigma\kappa}\theta_{\nu\beta}\theta^\kappa{}_\alpha = \partial^\sigma\epsilon_{\mu\sigma\kappa}\theta_{\nu\beta}\left(\delta^\kappa{}_\alpha-\frac{\partial^\kappa\partial_\alpha}{\Box}\right) = \partial^\sigma\epsilon_{\mu\sigma\alpha}\theta_{\nu\beta}. \tag{3.3.20}
$$

Assim,

$$
S_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{\partial^\sigma}{2}\left(\epsilon_{\mu\sigma\alpha}\theta_{\nu\beta}+\epsilon_{\mu\sigma\beta}\theta_{\nu\alpha}+\epsilon_{\nu\sigma\alpha}\theta_{\mu\beta}+\epsilon_{\nu\sigma\beta}\theta_{\mu\alpha}\right) = S_{\mu\nu,\alpha\beta}. \tag{3.3.21}
$$

A outra contração não nula é apresentada a seguir (detalhes no apêndice D):

$$
S_{\mu\nu,\kappa\lambda}S^{\kappa\lambda}{}_{,\alpha\beta} = -4\Box P^{(2)}_{\mu\nu,\alpha\beta} \tag{3.3.22}
$$

$$
\begin{aligned}
S_{\mu\nu,\kappa\lambda}S^{\kappa\lambda}{}_{,\alpha\beta} &= -2\Box\left(P^{(0-\theta)}+P^{(2)}\right)+\Box\left(2P^{(0-\theta)}-2P^{(2)}\right) \\
&= -4\Box P^{(2)}.
\end{aligned} \tag{3.3.23}
$$

Note que o processo de calcular as contrações entre os operadores de Barnes-Rivers e $S_{\mu\nu,\alpha\beta}$ não conduziu a uma estrutura nova, por exemplo outras combinações entre $S_{\mu\nu}$ e $\theta_{\mu\nu}$ ou com $\omega_{\mu\nu}$. Isso significa que a álgebra está completa apenas com a introdução de $S_{\mu\nu,\alpha\beta}$. É importante fazer esse comentário porque caso surgisse termos diferentes dos já conhecidos, seria necessário a inclusão dessas novas estruturas como operadores independentes e a repetição do cálculo das contrações até que isso não mais ocorresse. Finalmente, completamos a tabela com a álgebra dos operadores de Barnes-Rivers estendida para a gravitação de Chern-Simons, exibida na tabela (3.1).

3.3.2 Cálculo do propagador do gráviton em (2+1) dimensões

Após a obtenção da base a ser utilizada num cenário em (2+1) dimensões, vamos implementar o algoritmo já estudado no capítulo anterior. Desse modo, escrevemos os operadores $\mathcal{O}_{\mu\nu,\alpha\beta}$ e $\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}$ como uma combinação linear dos projetores (3.3.12-3.3.17), ou seja:

$$
\mathcal{O}_{\mu\nu,\alpha\beta} = a_1P^{(1)}+a_2P^{(2)}+a_3P^{(0-\theta)}+a_4P^{(0-\omega)}+a_5P^{(0-\theta\omega)}+a_6S, \tag{3.3.24}
$$

e

$$
\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = b_1P^{(1)}+b_2P^{(2)}+b_3P^{(0-\theta)}+b_4P^{(0-\omega)}+b_5P^{(0-\theta\omega)}+b_6S. \tag{3.3.25}
$$

	$P^{(1)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(2)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}$	$P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta}$	$S^{\kappa\lambda}{}_{,\alpha\beta}$
$P^{(1)}_{\mu\nu,\kappa\lambda}$	$P^{(1)}_{\mu\nu,\alpha\beta}$	0	0	0	0	0
$P^{(2)}_{\mu\nu,\kappa\lambda}$	0	$P^{(2)}_{\mu\nu,\alpha\beta}$	0	0	0	$P_{\mu\nu,\alpha\beta}$
$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}$	0	0	$P^{(0-\theta)}_{\mu\nu,\alpha\beta}$	0	$\frac{1}{\sqrt{2}}\theta_{\mu\nu}\omega_{\alpha\beta}$	0
$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}$	0	0	0	$P^{(0-\omega)}_{\mu\nu,\alpha\beta}$	$\frac{1}{\sqrt{2}}\omega_{\mu\nu}\theta_{\alpha\beta}$	0
$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}$	0	0	$\frac{1}{\sqrt{2}}\theta_{\mu\nu}\omega_{\alpha\beta}$	$\frac{1}{\sqrt{2}}\omega_{\mu\nu}\theta_{\alpha\beta}$	$P^{(0-\theta)}_{\mu\nu,\alpha\beta}+P^{(0-\omega)}_{\mu\nu,\alpha\beta}$	0
$S_{\mu\nu,\kappa\lambda}$	0	$S_{\mu\nu,\alpha\beta}$	0	0	0	$-4\Box P^{(2)}_{\mu\nu,\alpha\beta}$

Tabela 3.1: Álgebra dos operadores de Barnes-Rivers estendida

Seguindo o mesmo procedimento, vamos obter a forma geral do propagador do gráviton em (2+1) dimensões. A contração dos operadores (3.3.24) e (3.3.25) gera a identidade

$$\mathcal{I} = \left(a_1P^{(1)} + a_2P^{(2)} + a_3P^{(0-\theta)} + a_4P^{(0-\omega)} + a_5P^{(0-\theta\omega)} + a_6S\right) \times \\ \left(b_1P^{(1)} + b_2P^{(2)} + b_3P^{(0-\theta)} + b_4P^{(0-\omega)} + b_5P^{(0-\theta\omega)} + b_6S\right).$$

Desenvolvendo e usando os dados da tabela (3.1), encontramos

$$\mathcal{I} = a_3\left(b_3P^{(0-\theta)} + \frac{1}{\sqrt{2}}b_5P^{(\theta\omega)}\right) + a_1b_1P^{(1)} + a_2b_2P^{(2)} + a_4\left(b_4P^{(0-\omega)} + \frac{1}{\sqrt{2}}b_5P^{(\omega\theta)}\right) \\ + a_5\left[\frac{1}{\sqrt{2}}b_3P^{(\omega\theta)} + \frac{1}{\sqrt{2}}b_4P^{(\theta\omega)} + b_5\left(P^{(0-\theta)} + P^{(0-\omega)}\right)\right] \\ + \left(a_2b_6 + a_6b_2\right)S - 4a_6b_6\Box P^{(2)}.$$

Agrupando os termos semelhantes e escrevendo a identidade como (2.2.30), temos:

$$P^{(1)}_{\mu\nu,\alpha\beta} + P^{(2)}_{\mu\nu,\alpha\beta} + P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} = \left(a_3b_3 + a_5b_5\right)P^{(0-\theta)} + a_1b_1P^{(1)} + \left(a_2b_2 - 4a_6b_6\Box\right)P^{(2)} \\ + \left(a_4b_4 + a_5b_5\right)P^{(0-\omega)} + \frac{1}{\sqrt{2}}\left(a_3b_5 + a_5b_4\right)P^{(\theta\omega)} \\ + \frac{1}{\sqrt{2}}\left(a_4b_5 + a_5b_3\right)P^{(\omega\theta)} + \left(a_2b_6 + a_6b_2\right)S.$$

Ao compararmos os coeficientes, obtemos o seguinte sistema de equações

$$a_3b_3 + a_5b_5 = 1, \tag{3.3.26}$$
$$a_1b_1 = 1, \tag{3.3.27}$$
$$a_2b_2 - 4a_6b_6\Box = 1, \tag{3.3.28}$$
$$a_4b_4 + a_5b_5 = 1, \tag{3.3.29}$$
$$a_3b_5 + a_5b_4 = 0, \tag{3.3.30}$$
$$a_4b_5 + a_5b_3 = 0, \tag{3.3.31}$$
$$a_2b_6 + a_6b_2 = 0. \tag{3.3.32}$$

Isolando b_6 em (3.3.32) e substituindo em (3.3.28), encontramos

$$b_2 = \frac{a_2}{a_2^2 + 4a_6^2\Box}. \tag{3.3.33}$$

Substituindo o valor obtido para b_2 em (3.3.32), resulta

$$b_6 = -\frac{a_6}{a_2^2 + 4a_6^2\Box}. \tag{3.3.34}$$

A solução total do sistema é dada a seguir:

$$b_1 = \frac{1}{a_1},\ b_2 = \frac{a_2}{a_2^2 + 4a_6^2\Box},\ b_3 = -\frac{a_4}{a_5^2 - a_3 a_4}, \tag{3.3.35}$$

$$b_4 = -\frac{a_3}{a_5^2 - a_3 a_4},\ b_5 = \frac{a_5}{a_5^2 - a_3 a_4},\ b_6 = -\frac{a_6}{a_2^2 + 4a_6^2\Box}. \tag{3.3.36}$$

Portanto, a forma geral do propagador do gráviton em (2+1) dimensões é proporcional à estrutura,

$$\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = \frac{1}{a_1}P^{(1)} - \frac{1}{a_5^2 - a_3 a_4}\left(a_4 P^{(0-\theta)} + a_3 P^{(0-\omega)} - a_5 P^{(0-\theta\omega)}\right) + \frac{1}{a_2^2 + 4a_6^2\Box}\left(a_2 P^{(2)} - a_6 S\right). \tag{3.3.37}$$

Precisamos agora saber quem sçao os coeficientes a_i que caracterizam a lagrangiana (3.3.1). Lembramos que a lagrangiana de Einstein-Hilbert é a mesma em (2+1) dimensões. Logo, podemos "aproveitar" o resultado (2.3.36), que representa o operador quadrático associado ao setor de Einstein-Hilbert. Também temos que adicionar a contribuição do termo de Chern-Simons, dado na eq. (3.3.9). Escrevemos assim:

$$\begin{aligned}\mathcal{O}_{\mu\nu,\alpha\beta} &= \frac{1}{2}\left(2P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{2}P^{(0-\theta\omega)}\right)\Box - \frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{2}P^{(0-\theta\omega)}\right)\Box \\ &\quad - \frac{1}{2}\left(P^{(0-\theta)} + P^{(0-\omega)} + P^{(1)} + P^{(2)}\right)\Box + \frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right)\Box \\ &\quad + \frac{1}{\alpha}\left[\frac{1}{2}\left(2P^{(0-\omega)} + \sqrt{2}P^{(0-\theta\omega)}\right) - \frac{1}{2}\left(P^{(1)} + 2P^{(0-\omega)}\right)\right. \\ &\quad \left. - \frac{1}{4}\left(2P^{(0-\theta)} + P^{(0-\omega)} + \sqrt{2}P^{(0-\theta\omega)}\right)\right]\Box + \frac{\Box}{4\tau}S.\end{aligned}$$

Agrupando os termos semelhantes, temos

$$\mathcal{O}_{\mu\nu,\alpha\beta} = \left(\frac{\alpha - 1}{2\alpha}\right)\Box P^{(0-\theta)} - \frac{\Box}{2\alpha}P^{(1)} - \frac{\Box}{2}P^{(2)} - \frac{\Box}{4\alpha}P^{(0-\omega)} + \frac{\sqrt{2}\Box}{4\alpha}P^{(0-\theta\omega)} + \frac{\Box}{4\tau}S. \tag{3.3.38}$$

Por meio desse procedimento, encontramos

$$a_1 = -\frac{\Box}{2\alpha},\ a_2 = -\frac{\Box}{2},\ a_3 = \left(\frac{\alpha - 1}{2\alpha}\right)\Box,\ a_4 = -\frac{\Box}{4\alpha},\ a_5 = \frac{\sqrt{2}\Box}{4\alpha}\ \text{e}\ a_6 = \frac{\Box}{4\tau}, \tag{3.3.39}$$

o que permite escrever os coeficientes b_i a seguir

$$b_1 = -\frac{2\alpha}{\Box},\ b_2 = -\frac{2\tau^2}{\tau^2\Box - 4\Box^2},\ b_3 = \frac{2}{\Box},$$

$$b_4 = \frac{4}{\Box}(1 - \alpha),\ b_5 = \frac{2}{\Box},\ \text{e}\ b_6 = -\frac{\tau}{\tau^2\Box - 4\Box^2}.$$

Podemos então escrever explicitamente o propagador do gráviton (no espaço dos momentos):

$$\Delta_{\mu\nu,\alpha\beta} = i\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} = \frac{i}{p^2}\left\{2\alpha P^{(1)}_{\mu\nu,\alpha\beta} + 4\left[(\alpha-1)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \frac{1}{2}\left(P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \sqrt{2}P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\right)\right]\right.$$
$$\left. + \frac{2\tau^2}{\tau^2-4p^2}\left(P^{(2)}_{\mu\nu,\alpha\beta} - \frac{1}{2\tau}S_{\mu\nu,\alpha\beta}\right)\right\}. \quad (3.3.40)$$

Além do polo usual ($p^2 = 0$), temos um outro que surge da própria estrutura do espaço-tempo de (2+1) dimensões, que fornece a seguinte relação de dispersão

$$\tau^2 - 4p^2 = 0. \quad (3.3.41)$$

Ao isolar p_0 no primeiro membro, temos

$$p_0 = \pm\sqrt{|\mathbf{p}|^2 + \frac{\tau^2}{4}}, \quad (3.3.42)$$

que é a relação de dispersão de um gráviton massivo, e consistente com os requisitos de causalidade. De fato,

$$u_g = \frac{|\mathbf{p}|}{\sqrt{|\mathbf{p}|^2 + \frac{\tau^2}{4}}} < 1,$$

é a velocidade de grupo obtida para esse polo.

3.3.3 Unitariedade tree-level

Neste momento vamos analisar a unitariedade das excitações do propagador (3.3.40). Antes de iniciarmos tal discussão, devemos saber se o projetor $S_{\mu\nu,\alpha\beta}$ contribuirá ou não para a saturação, temos

$$\begin{aligned}
J^{\mu\nu}S_{\mu\nu,\alpha\beta}J^{\alpha\beta} &= \frac{1}{2}J^{\mu\nu}\left(S_{\mu\alpha}\theta_{\nu\beta} + S_{\mu\beta}\theta_{\nu\alpha} + S_{\nu\alpha}\theta_{\mu\beta} + S_{\nu\beta}\theta_{\mu\alpha}\right)J^{\alpha\beta}, \\
&= \frac{1}{2}J^{\mu\nu}\left(S_{\mu\alpha}\eta_{\nu\beta} + S_{\mu\beta}\eta_{\nu\alpha} + S_{\nu\alpha}\eta_{\mu\beta} + S_{\nu\beta}\eta_{\mu\alpha}\right)J^{\alpha\beta}, \\
&= \frac{1}{2}\left(S_{\mu\alpha}J^{\mu}{}_{\beta} + S_{\mu\beta}J^{\mu}{}_{\alpha} + S_{\nu\alpha}J^{\nu}{}_{\beta} + S_{\nu\beta}J^{\nu}{}_{\alpha}\right)J^{\alpha\beta}, \\
&= \left(S_{\mu\alpha}J^{\mu}{}_{\beta} + S_{\mu\beta}J^{\mu}{}_{\alpha}\right)J^{\alpha\beta}, \\
&= 2S_{\mu\alpha}J^{\mu}{}_{\beta}J^{\alpha\beta},
\end{aligned}$$

note que

$$\begin{aligned}
S_{\mu\alpha}J^{\mu}{}_{\beta}J^{\alpha\beta} &= -S_{\alpha\mu}J^{\alpha\beta}J^{\mu}{}_{\beta}, \\
&= -S_{\alpha\mu}J^{\alpha}{}_{\beta}J^{\mu\beta}, \\
&= -S_{\mu\alpha}J^{\mu}{}_{\beta}J^{\alpha\beta},
\end{aligned}$$

logo

$$S_{\mu\alpha}J^{\mu}{}_{\beta}J^{\alpha\beta} = 0, \quad (3.3.43)$$

e assim a saturação de $P_{\mu\nu,\alpha\beta}$ por correntes externas não deverá contribuir,

$$J^{\mu\nu}S_{\mu\nu,\alpha\beta}J^{\alpha\beta} = 0. \quad (3.3.44)$$

Importante observar também que os resultados para a saturação dos outros dois projetores deve ser corrigido para o presente cenário,

$$J^{\mu\nu}P^{(2)}_{\mu\nu,\alpha\beta}J^{\alpha\beta} = J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{2}\left(J^{\alpha}{}_{\alpha}\right)^2, \tag{3.3.45}$$

$$J^{\mu\nu}P^{(0-\theta)}_{\mu\nu,\alpha\beta}J^{\alpha\beta} = \frac{1}{2}\left(J^{\alpha}{}_{\alpha}\right)^2. \tag{3.3.46}$$

Usando o propagador (3.3.40), temos

$$\begin{aligned} SP &= J^{\mu\nu}Res\left[i\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}\right]J^{\alpha\beta}, \\ &= -iRes\left[\frac{2}{p^2}\left(\frac{\tau^2}{4}\frac{J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{2}\left(J^{\alpha}{}_{\alpha}\right)^2}{p^2 - \frac{\tau^2}{4}} + \frac{1}{2}\left(J^{\alpha}{}_{\alpha}\right)^2\right)\right]. \end{aligned}$$

O resíduo do primeiro termo em $p^2 = 0$ será dado por:

$$\begin{aligned} Res|_{p^2=0} &= -2\left[J_{\alpha\beta}J^{\alpha\beta} + \frac{1}{2}\left(J^{\alpha}{}_{\alpha}\right)^2\right], \\ &= -\left[\left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2} + J_{aa}\right)^2 + 2\left(J_{ca}\right)^2 - 4\frac{\left(p_c J_{ca}\right)^2}{|\mathbf{p}|} - 2\left(J_{aa}\right)^2\right], \end{aligned}$$

que ao compararmos com o resultado (2.4.17), implica em uma contribuição de norma negativa. Portanto, tal modo é não unitário. Para o polo $p^2 - \frac{\tau^2}{4} = 0$, temos

$$\begin{aligned} Res|_{p^2-\frac{\tau^2}{4}=0} &= 2\left(J_{\alpha\beta}J^{\alpha\beta} - \frac{1}{2}\left(J^{\alpha}{}_{\alpha}\right)^2\right) \\ &= \left(\frac{p_a p_c J_{ca}}{|\mathbf{p}|^2 + \frac{\tau^2}{4}} + J_{aa}\right)^2 + 2\left(J_{ca}\right)^2 - 4\frac{\left(p_c J_{ca}\right)^2}{\sqrt{|\mathbf{p}|^2 + \frac{\tau^2}{4}}} - 2\left(J_{aa}\right)^2. \end{aligned} \tag{3.3.47}$$

Este segundo modo, ao contrário do anterior, está associado com excitações de norma positiva e isso nos permite chegar a conclusão de que o mesmo é unitário. Entretanto, pelo fato de um dos modos violar a unitariedade, a gravitação Einstein-Chern-Simons como um todo não é unitária.

CAPÍTULO 4

Gravitação de Einstein-Hilbert modificada por termo de violação da simetria de Lorentz.

Nos capítulos precedentes, revisitamos o cálculo do propagador do gráviton em cenários bem distintos. Vamos agora examinar a obtenção do propagador do gráviton em uma teoria de campo em que há violação da simetria de Lorentz, ou seja, um teoria de campo além do Modelo Padrão (MP) [22].

O MP é uma teoria que se propõe a descrever tanto as interações fundamentais forte, fraca e eletromagnética, que atuam entre as partículas elementares (férmions e bósons). Apesar de obter grande sucesso, como a comprovação da existência do Bóson de Higgs, não podemos nos referir ao MP como uma teoria completa no contexto da física de partículas porque ainda existem algumas questões em aberto, tais como: onde a gravidade se encaixa no MP? Qual a origem da massa dos neutrinos? Por que existe assimetria entre matéria e antimatéria? Qual a origem e essência da matéria escura? Essas pergunta ainda não possuem respostas, sugerindo a possível existência de teorias para as interações fundamentais que vão além do Modelo Padrão. Entre estas, situa-se o Modelo Padrão Estendido (MPE) [23-32] como uma possibilidade.

O MPE, por ser uma extensão do MP, contém todas as interações por este descritas, incluindo novos termos de interação em cada um dos seus setores. Esses novos termos que violam a simetria de Lorentz e CPT (no referencial das partículas), são compostos por contrações tensoriais entre os campos físicos da teoria e campos de fundo (fixos), que carregam a informação da violação da simetria de Lorentz [23-33]. A ideia por trás dos mecanismos de violação de simetria é justamente a busca de uma teoria mais fundamental, válida em altíssimas energias (na escala de Planck). Se a simetria de Lorentz é violada espontaneamente em altas energias, gera-se quantidades esperadas no vácuo não-nulas, que funcionam como efeito remanescente da quebra. Não poderíamos deixar de mencionar que uma teoria que viola a simetria CPT, inevitavelmente irá violar a simetria de Lorentz [34]. Em contrapartida, preservar a simetria CPT não nos dá garantia alguma de que não ocorra violação da simetria de Lorentz fora do cone de luz, ou seja, pode ser que aconteça, mas sem qualquer conexão causal.

O processo da quebra de simetria pode ocorrer de duas principais formas: espontânea ou explícita [35]. A principal diferença entre esses dois processos reside na forma que os termos de violação surgem. Na primeira eles se originam por meio do mecanismo de quebra espontânea (como o mecanismo de Higgs). Na segunda os termos de violação se acoplam diretamente aos campos físicos na lagrangiana de alguma teoria, ou seja, os campos de fundo fixos são introduzidos à mão "forçando" a violação da simetria de Lorentz. Este trabalho contempla o caso da violação espontânea de simetria no setor gravitacional, sendo estudado por meio de um formalismo que mantenha a invariância sob transformação de coordenadas no referencial do observador.

Desde que as teorias de campo que violam a simetria de Lorentz foram sugeridas como uma extensão ao MP vigente, teóricos em várias partes do mundo se interessaram em investigar a fundo as implicações da quebra de simetria nos diferentes setores do MP e compará-los com os resultados já conhecidos.

Em 2004 [41], Kostelecky propôs um modelo teórico no qual os termos de violação de Lorentz foram introduzidos por meio de tetradas e conexões de spin dentro da geometria de Riemann-Cartan provida de torsão e dinâmica de curvatura. No ano seguinte, um novo estudo [42] foi desenvolvido usando o formalismo apresentado no trabalho do ano anterior. Neste artigo, foi analisado a conexão entre os modos de Nambu-Goldstone e a violação espontânea de simetria de Lorentz e difeomorfismo. Este capítulo traz alguns resultados obtidos em [43], que foi o precursor de outras investigações que abordaram essa questão [43-52], os resultados obtidos de tais investigações sugerem interessantes direcionamentos para mais desenvolvimentos teóricos e testes experimentais. Para exemplificar este fato, podemos citar o trabalho de A. Cavalcante *et.al*[53], onde foi obtida uma solução do tipo Schwarszchild para o mesmo modelo de gravidade abordado nesse capítulo, podendo ser testada por meio da precessão do periélio de planetas, curvatura da luz e efeitos de atraso temporal.

4.1 Modelo teórico

A investigação do setor gravitacional do MPE tem sido amplamente realizada em diversos trabalhos recentes [41-53]. O foco dessa seção é o estudo do propagador do gráviton no contexto de um modelo gravitacional que inclui termos de violação de Lorentz. Neste sentido, vamos considerar o chamado modelo "bumblebee", constituído por um campo vetorial no formalismo da relatividade geral para uma teoria de campo na presença do termos de violação da simetria de Lorentz.

Como descrito anteriormente, sob a ação de um potencial apropriado, o campo bumblebee B_μ adquire um valor esperado de vácuo não nulo b_μ, o que induz a violação espontânea da simetria de Lorentz no setor gravitacional do MPE. A ação total deste modelo consiste na ação de Einstein-Hilbert somada à ação do termo de violação de simetria de Lorentz, ou seja,

$$\mathcal{S} = \mathcal{S}_{EH} + \mathcal{S}_{LV}, \tag{4.1.1}$$

onde $\mathcal{L}_{EH}$ é dada na eq. (2.3.11) e $\mathcal{L}_{LV}$ representa as contribuições da violação de Lorentz, dada a seguir:

$$\mathcal{S}_{LV} = \int d^4x\sqrt{-g}\frac{2}{\kappa^2}\left(uR + s^{\mu\nu}R_{\mu\nu} + t^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}\right). \tag{4.1.2}$$

Aqui, u representa um escalar, enquanto $s^{\mu\nu}$ e $t^{\mu\nu\alpha\beta}$ são tensores adimensionais que contêm os coeficientes de violação da simetria de Lorentz, possuindo as mesmas simetrias dos tensores de Ricci e Riemann. Por fim, κ^2 é a conhecida constante de acoplamento gravitacional. Podemos salientar também que estes tensores possuem traço-nulo, uma vez que $s^\mu{}_\mu$ e $t^{\mu\nu}{}_{\mu\nu}$ contribuem na lagrangiana com termos proporcionais a R, pois:

$$s^{\mu\nu}R_{\mu\nu} = s^\mu{}_\nu R^\nu{}_\mu = 2s^0{}_i R^i{}_0 + \underbrace{s^i{}_j R^j{}_i}_{\text{com } i\neq j} + s^\mu{}_\mu R^\mu{}_\mu. \tag{4.1.3}$$

Dessa forma, o traço de $s^{\mu\nu}$ contribui na ação com termo proporcional a R, que pode ser absorvida no termo uR. O mesmo raciocínio se aplica ao traço do tensor $t^{\mu\nu\alpha\beta}$. Formalmente, podemos tomar $t^{\mu\nu\alpha\beta} = 0$, e parametrizar as quantidades u e $s^{\mu\nu}$ em termos de um campo vetorial B^{μ}. Escrevemos assim:

$$u = \frac{1}{4}\xi B^{\alpha}B_{\alpha}, \tag{4.1.4}$$

$$s^{\mu\nu} = \xi\left(B^{\mu}B^{\nu} - \frac{1}{4}g^{\mu\nu}B^{\alpha}B_{\alpha}\right), \tag{4.1.5}$$

onde ξ tem dimensão de massa $[\xi] = -2$, o que mantém adimensionais u e $s^{\mu\nu}$, conforme [41]. O campo vetorial B^{μ} é chamado de campo "bumblebee". A constante ξ é quem estabelece o acoplamento entre o campo bumblebee e o tensor de curvatura do espaço-tempo. A dinâmica do campo bumblebee descrita por uma ação que congrega os acoplamentos (4.1.4) e (4.1.5) e o termo dinâmico:

$$\mathcal{S}_B = \int d^4x\sqrt{-g}\left[-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{2\xi}{\kappa^2}B^{\mu}B^{\nu}R_{\mu\nu} - V\left(B_{\mu}B^{\mu} \mp b^2\right)\right], \tag{4.1.6}$$

onde $B_{\mu\nu}$ é o tensor antissimétrico que faz o papel de "field-strength" do campo bumblebee,

$$B_{\mu\nu} = \partial_{\mu}B_{\nu} - \partial_{\nu}B_{\mu}. \tag{4.1.7}$$

Na ação (4.1.6) temos ainda o potencial,

$$V = \frac{\lambda}{2}\left(B^{\mu}B_{\mu} \mp b^2\right)^2, \tag{4.1.8}$$

causador da quebra espontânea da simetria de difeomorfismo associada à quebra espontânea deflagrada no setor do campo bumblebee. Aqui, λ é adimensional e b^2 é uma constante positiva relacionada ao valor esperado de vácuo não-nulo do campo bumblebee.

É importante ainda mencionar que a ação (4.1.1) deixa de ser invariante sob a transformação,

$$h_{\mu\nu}(x) \to h_{\mu\nu}(x) - \partial_{\nu}\xi_{\mu}(x) - \partial_{\mu}\xi_{\nu}(x), \tag{4.1.9}$$

que se deve à presença do campo bumblebee neste modelo.

4.1.1 Linearização da lagrangiana $\mathcal{L}_{LV}$

A fim de estudar os efeitos da presença do campo bumblebee sobre o propagador do gráviton, vamos buscar uma forma linearizada e quadrática em $h^{\mu\nu}$ da lagrangiana $\mathcal{L}_{LV}$. Partindo da eq. [4.1.2] com $t^{\mu\nu\alpha\beta} = 0$, resta:

$$\mathcal{L}_{LV} = \sigma\sqrt{-g}B^{\mu}B^{\nu}R_{\mu\nu}, \tag{4.1.10}$$

sendo $\sigma = 2\xi/\kappa^2$. O cálculo do propagador, como visto nos capítulos anteriores, requer uma expansão da lagrangiana em segunda ordem nos campos propagantes. Assim, devemos escrever (4.1.10) como função de termos em segunda ordem de $h^{\mu\nu}$. Podemos executar essa tarefa escrevendo as excitações dos campos em torno dos valores esperados não-nulos (b^{μ}) através de pequenas flutuações ($h^{\mu\nu}$ e $\tilde{B}^{\mu}$), como dado a seguir:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu}, \tag{4.1.11}$$

$$B_{\mu} = b_{\mu} + \tilde{B}_{\mu}, \tag{4.1.12}$$

$$B^{\mu} = b^{\mu} + \tilde{B}^{\mu} - \kappa b_{\nu}h^{\mu\nu}. \tag{4.1.13}$$

Note que a única barreira que nos impede de escrever diretamente a expansão da eq. (4.1.10) em segunda ordem é o desconhecimento de uma expressão para $\tilde{B}^\mu$ e, para determiná-la, vamos obter as equações de movimento a partir da ação

$$\mathcal{S}_B = \int d^4x\sqrt{-g}\left[-\frac{1}{4}B_{\mu\nu}B^{\mu\nu} + \frac{2\xi}{\kappa^2}B^\mu B^\nu R_{\mu\nu} - V\left(B_\mu B^\mu \mp b^2\right)\right], \tag{4.1.14}$$

e resolvê-la para $\tilde{B}^\mu$ com o potencial específico

$$V = \frac{\lambda}{2}\left(B^\mu B_\mu \mp b^2\right)^2. \tag{4.1.15}$$

Usaremos as equações de Euler-Lagrange para o campo bumblebee

$$\frac{\partial \mathcal{L}_B}{\partial B_\mu} - \partial_\nu\left[\frac{\partial \mathcal{L}_B}{\partial\left(\partial_\nu B_\mu\right)}\right] = 0, \tag{4.1.16}$$

nos mesmos moldes utilizados para um campo vetorial usual. Para a primeira parcela,

$$\frac{\partial \mathcal{L}_B}{\partial B_\mu} = \frac{\partial}{\partial B_\mu}\left[-\frac{1}{4}B_{\alpha\beta}B^{\alpha\beta} + \sigma B^\alpha B^\beta R_{\alpha\beta} - V\left(B_\alpha B^\alpha \mp b^2\right)\right], \tag{4.1.17}$$

encontramos:

$$\frac{\partial \mathcal{L}_B}{\partial B_\mu} = 2\sigma B_\alpha R^{\mu\alpha} - 2\lambda\left(B_\alpha B^\alpha \mp b^2\right)B^\mu. \tag{4.1.18}$$

A segunda parcela da equação de Euler-Lagrange resume-se a

$$\frac{\partial \mathcal{L}_B}{\partial\left(\partial_\nu B_\mu\right)} = -\frac{1}{4}\frac{\partial}{\partial\left(\partial_\nu B_\mu\right)}\left(B_{\alpha\beta}B^{\alpha\beta}\right) = -\frac{1}{4}\left(-4B^{\mu\nu}\right) = B^{\mu\nu}, \tag{4.1.19}$$

logo,

$$\partial_\nu\left[\frac{\partial \mathcal{L}_B}{\partial\left(\partial_\nu B_\mu\right)}\right] = \partial_\nu B^{\mu\nu}. \tag{4.1.20}$$

Substituindo tais resultados na eq. (4.1.16), obtemos a equação de movimento

$$2\sigma B_\alpha R^{\mu\alpha} - 2\lambda\left(B_\alpha B^\alpha \mp b^2\right)B^\mu = \partial_\nu B^{\mu\nu}. \tag{4.1.21}$$

Na equação acima, podemos aplicar a aproximação adotada nas eqs. $(4.1.11 - 4.1.13)$ e simplificar os termos para obter:

$$\left(\Box\eta_{\mu\nu} - \partial_\mu\partial_\nu - 4\lambda b_\mu b_\nu\right)\tilde{B}^\mu = -2\lambda\kappa b_\nu b_\alpha b_\beta h^{\alpha\beta} - 2\sigma b^\alpha R_{\alpha\nu}, \tag{4.1.22}$$

que no espaço dos momentos pode ser lida como

$$\begin{aligned}\left(-p^2\eta_{\mu\nu} + p_\mu p_\nu - 4\lambda b_\mu b_\nu\right)\tilde{B}^\mu &= -2\lambda\kappa b_\nu b_\alpha b_\beta h^{\alpha\beta} - 2\sigma b^\alpha R_{\alpha\nu},\\ \mathcal{O}_{\mu\nu}\tilde{B}^\mu &= J_\nu,\end{aligned} \tag{4.1.23}$$

onde definimos,

$$\mathcal{O}_{\mu\nu} = -p^2\eta_{\mu\nu} + p_\mu p_\nu - 4\lambda b_\mu b_\nu, \qquad J_\nu = -2\lambda\kappa b_\nu b_\alpha b_\beta h^{\alpha\beta} - 2\sigma b^\alpha R_{\alpha\nu}, \tag{4.1.24}$$

que satisfaz a identidade

$$\mathcal{O}_{\mu\alpha}\Delta^{\alpha\nu} = \delta^\nu{}_\mu, \tag{4.1.25}$$

onde ${\delta^{\nu}}_{\mu}$ é o delta de Kronecker. A eq. (4.1.23) pode ser solucionada pelo método de Green. Para tal, a contraímos com o operador inverso de $\theta^{\alpha\nu}$, que satisfaz (4.1.25). Realizando a contração da eq. (4.1.23) com o operador $\Delta^{\alpha\nu}$, temos :

$$\Delta^{\alpha\nu}\mathcal{O}_{\mu\nu}\tilde{B}^{\mu} = \Delta^{\alpha\nu}J_{\nu}, \tag{4.1.26}$$

$$\tilde{B}^{\alpha} = \Delta^{\alpha\nu}J_{\nu}. \tag{4.1.27}$$

A fim de determinar $\Delta^{\alpha\nu}$, vamos usar o método de projetores tensoriais, expandindo-o na base de projetores $[\eta_{\alpha\beta}, p_{\alpha}p_{\beta}, b_{\alpha}b_{\beta}, (p_{\alpha}b_{\beta} + p_{\beta}b_{\alpha})]$, ou seja,

$$\Delta_{\alpha\beta} = a_1\eta_{\alpha\beta} + a_2 p_{\alpha}p_{\beta} + a_3 b_{\alpha}b_{\beta} + a_4(p_{\alpha}b_{\beta} + p_{\beta}b_{\alpha}), \tag{4.1.28}$$

onde a_1, a_2, a_3 e a_4 são coeficientes a serem determinados. Implementando (4.1.28) na eq. (4.1.25), temos:

$$\left(-p^2\eta_{\mu\alpha} + p_{\mu}p_{\alpha} - 4\lambda b_{\mu}b_{\alpha}\right)\left[a_1{\delta^{\alpha}}_{\nu} + a_2 p^{\alpha}p_{\nu} + a_3 b^{\alpha}b_{\nu} + a_4(p^{\alpha}b_{\nu} + p_{\nu}b^{\alpha})\right] = \eta_{\mu\nu}. \tag{4.1.29}$$

Desenvolvendo os produtos e realizando as simplificações devidas, resulta:

$$\begin{aligned}
&- p^2\left[a_1\eta_{\mu\nu} + a_2 p_{\mu}p_{\nu} + a_3 b_{\mu}b_{\nu} + a_4(p_{\mu}b_{\nu} + p_{\nu}b_{\mu})\right] \\
&\qquad + \left\{a_1 p_{\mu}p_{\nu} + a_2 p^2 p_{\mu}p_{\nu} + a_3\left(b\cdot p\right)p_{\mu}b_{\nu} + a_4\left[\left(p^2 p_{\mu}b_{\nu} + \left(b\cdot p\right)p_{\mu}p_{\nu}\right]\right\}\right. \\
&\qquad\quad - 4\lambda\left\{a_1 b_{\mu}b_{\nu} + a_2\left(b\cdot p\right)p_{\nu}b_{\mu} + a_3 b^2 b_{\nu}b_{\mu} + a_4\left[\left(\left(b\cdot p\right)b_{\nu}b_{\mu} + b^2 p_{\nu}b_{\mu}\right)\right]\right\} = \eta_{\mu\nu}.
\end{aligned}$$

Agrupamos os termos semelhantes, obtemos o seguinte sistema:

$$\begin{aligned}
-a_1p^2 &= 1, \\
a_1 - a_2p^2 + a_2p^2 + a_4\left(b\cdot p\right) &= 0, \\
-a_3p^2 - 4\lambda a_1 - 4\lambda b^2 a_3 - 4\lambda\left(b\cdot p\right)a_4 &= 0, \\
-p^2 a_4 + a_3\left(b\cdot p\right) + p^2 a_4 &= 0, \\
-p^2 a_4 - 4\lambda\left(b\cdot p\right)a_2 - 4\lambda b^2 a_4 &= 0,
\end{aligned} \tag{4.1.30}$$

cuja solução conduz aos coeficientes a_i a seguir:

$$a_1 = -\frac{1}{p^2}, \qquad a_4 = -\frac{a_1}{\left(b\cdot p\right)} = \frac{1}{p^2\left(b\cdot p\right)}, \tag{4.1.31}$$

$$a_3 = 0, \qquad a_2 = -\frac{\left(p^2 + 4\lambda b^2\right)}{4\lambda\left(b\cdot p\right)}a_4 = -\frac{\left(p^2 + 4\lambda b^2\right)}{4\lambda p^2\left(b\cdot p\right)^2}. \tag{4.1.32}$$

Assim, o operador $\Delta_{\alpha\beta}$ tem a forma,

$$\Delta_{\alpha\beta} = -\frac{1}{p^2}\eta_{\alpha\beta} - \frac{\left(p^2 + 4\lambda b^2\right)}{4\lambda p^2\left(b\cdot p\right)^2}p_{\alpha}p_{\beta} + \frac{1}{p^2\left(b\cdot p\right)}(p_{\alpha}b_{\beta} + p_{\beta}b_{\alpha}). \tag{4.1.33}$$

Podemos então encontrar a seguinte expressão para $\tilde{B}^{\mu}$:

$$\tilde{B}^{\mu} = \left[-\frac{1}{p^2}\eta^{\mu\nu} - \frac{\left(p^2 + 4\lambda b^2\right)}{4\lambda p^2\left(b\cdot p\right)^2}p^{\mu}p^{\nu} + \frac{1}{p^2\left(b\cdot p\right)}(p^{\mu}b^{\nu} + p^{\nu}b^{\mu})\right]\left(-2\lambda\kappa b_{\nu}b_{\alpha}b_{\beta}h^{\alpha\beta} - 2\sigma b^{\alpha}R_{\alpha\nu}\right). \tag{4.1.34}$$

Desenvolvendo os produtos e simplificando os termos, resulta:

$$\tilde{B}^{\mu} = \frac{\kappa p^{\mu} b_{\alpha} b_{\beta} h^{\alpha\beta}}{2\,(b\cdot p)} + \frac{2\sigma b_{\alpha} R^{\mu\alpha}}{p^2} - \frac{2\sigma p^{\mu} b_{\alpha} b_{\beta} R^{\alpha\beta}}{p^2\,(b\cdot p)} + \frac{\sigma p^{\mu} R}{4\lambda\,(b\cdot p)} - \frac{\sigma b^{\mu} R}{p^2} + \frac{\sigma b^2 p^{\mu} R}{p^2\,(b\cdot p)}. \qquad (4.1.35)$$

Agora vamos retomar a eq. (4.1.10), escrevendo-a como uma expansão em segunda ordem no campo $h^{\mu\nu}$,

$$\mathcal{L}_{LV} = \sigma\sqrt{-g} B_{\mu} B_{\nu} R^{\mu\nu} = \sigma\left(1 + \frac{1}{2}\kappa h\right) B_{\mu} B_{\nu} R^{\mu\nu}. \qquad (4.1.36)$$

Note que não levamos em consideração os termos de segunda ordem na expansão de $\sqrt{-g}$. Uma vez que o tensor de Ricci fornece contribuições em primeira ordem de $h^{\mu\nu}$ ou superior, os termos em segunda ordem de $\sqrt{-g}$ implicariam em contribuições de pelo menos terceira ordem em $h^{\mu\nu}$. Sendo assim, consideraremos a expansão até a segunda ordem do tensor de Ricci,

$$\begin{aligned} \mathcal{L}_{LV} &= \sigma\left(1 + \frac{1}{2}\kappa h\right)\left(b_{\mu} + \tilde{B}_{\mu}\right)\left(b_{\nu} + \tilde{B}_{\nu}\right)\left[R^{\mu\nu}(h) + R^{\mu\nu}\left(h^2\right)\right], \\ \mathcal{L}_{LV} &= \sigma\left[R^{\mu\nu}(h) + R^{\mu\nu}\left(h^2\right) + \frac{1}{2}\kappa h R^{\mu\nu}(h)\right]\left(b_{\mu} b_{\nu} + b_{\mu}\tilde{B}_{\nu} + b_{\nu}\tilde{B}_{\mu} + \tilde{B}_{\mu}\tilde{B}_{\nu}\right) + O\left(h^3\right). \end{aligned} \qquad (4.1.37)$$

Simplificando essa expressão, resulta:

$$\mathcal{L}_{LV} = \sigma\left[b_{\mu} b_{\nu} R^{\mu\nu}\left(h^2\right) + 2 b_{\mu}\tilde{B}_{\nu} R^{\mu\nu}(h) + \frac{1}{2}\kappa h b_{\mu} b_{\nu} R^{\mu\nu}(h)\right] + O\left(h^3\right). \qquad (4.1.38)$$

Para finalizar o processo, basta substituir as expressões já obtidas para $\tilde{B}_{\nu}$, $R^{\mu\nu}(h)$ e $R^{\mu\nu}(h^2)$,

$$\begin{aligned} R_{\mu\nu} &= R_{\mu\nu}(h) + R_{\mu\nu}\left(h^2\right) + O\left(h^3\right) \\ R_{\mu\nu} &= -2\kappa\partial_{[\mu}\mathcal{G}^{\gamma}{}_{\gamma]\nu} + 2\kappa^2\mathcal{G}^{\kappa}{}_{\nu[\mu}\mathcal{G}^{\alpha}{}_{\alpha]\kappa} + O\left(h^3\right), \end{aligned} \qquad (4.1.39)$$

obtidas no apêndice B. A contribuição de primeira ordem em $h^{\mu\nu}$ do tensor de Ricci, escrita no espaço dos momentos é:

$$R_{\mu\nu}(h) = \frac{\kappa}{2}\left(p^2 h_{\mu\nu} + p_{\mu} p_{\nu} h - p_{\mu} p^{\gamma} h_{\gamma\nu} - p_{\nu} p^{\gamma} h_{\gamma\mu}\right). \qquad (4.1.40)$$

Para a segunda ordem, temos

$$\begin{aligned} R_{\mu\nu}\left(h^2\right) = \frac{\kappa^2}{4}&\left[\left(p_{\mu} h^{\kappa\alpha} + p^{\alpha} h^{\kappa}{}_{\mu} - p^{\kappa} h^{\alpha}{}_{\mu}\right)\left(p_{\nu} h_{\kappa\alpha} + p_{\kappa} h_{\nu\alpha} - p_{\alpha} h_{\kappa\nu}\right)\right. \\ &- p_{\mu} h p_{\kappa} h^{\kappa}{}_{\nu} + p_{\nu} h p_{\kappa} h^{\kappa}{}_{\mu} + p^2 h h_{\mu\nu} + 2 p_{\mu} p_{\alpha} h^{\alpha}{}_{\kappa} h^{\kappa}{}_{\nu} \\ &\left. -2 p_{\nu} p_{\alpha} h^{\alpha}{}_{\kappa} h^{\kappa}{}_{\mu} - 2 p_{\kappa} p_{\alpha} h^{\kappa\alpha} h_{\mu\nu}\right]. \end{aligned} \qquad (4.1.41)$$

A forma linearizada da lagrangiana $\mathcal{L}_{LV}$ pode ser obtida ao substituir as eqs. $(4.1.40 - 4.1.41)$

na eq. (4.1.38), resultando em:

$$\begin{aligned}
\mathcal{L}_{LV} = \xi \Bigg\{ & p^2 b_\mu b_\nu \left(h^{\mu\nu} h + h^{\mu\alpha} h^\nu{}_\alpha\right) - \frac{1}{2} (b \cdot p)^2 \left(h^{\mu\nu} h_{\mu\nu} - h^2\right) \\
& - \left[b_\mu b_\nu p_\alpha p_\beta + \frac{1}{4} (b_\mu p_\nu + b_\nu p_\mu)(b_\alpha p_\beta + b_\beta p_\alpha) \right] \Bigg\} h^{\mu\nu} h^{\alpha\beta} \\
& + \frac{4\xi^2}{\kappa^2} \Bigg\{ \left[-2p^2 b_\mu b_\nu - 2b^2 p_\mu p_\nu + 2(b \cdot p)(b_\mu p_\nu + b_\nu p_\mu) - \frac{p^2 p_\mu p_\nu}{2\lambda} \right] h^{\mu\nu} h \\
& + \Bigg[2b_\mu b_\nu p_\alpha p_\beta - \frac{1}{4} (b_\mu p_\nu + b_\nu p_\mu)(b_\alpha p_\beta + b_\beta p_\alpha) + \frac{b^2 p_\mu p_\nu p_\alpha p_\beta}{p^2} \\
& - \frac{(b \cdot p) p_\mu p_\nu}{p^2} (b_\alpha p_\beta + b_\beta p_\alpha) + \frac{p_\mu p_\nu p_\alpha p_\beta}{4\lambda} \Bigg] h^{\mu\nu} h^{\alpha\beta} + \left[b^2 p^2 - (b \cdot p)^2 + \frac{p^4}{4\lambda} \right] h^2 \\
& + \left[p^2 b_\mu b_\nu - (b \cdot p)(b_\mu p_\nu + b_\nu p_\mu) + \frac{(b \cdot p)^2 p_\mu p_\nu}{p^2} \right] h^{\mu\lambda} h^\nu{}_\lambda \Bigg\} . \qquad (4.1.42)
\end{aligned}$$

Se formos comparar a eq. (4.1.42) com a da ref. [22], iremos notar que o último termo da terceira linha está diferente, isso se justifica pelo fato de haver um pequeno “misprint” nesse artigo. A fim de obter a forma bilinear da lagrangiana (4.1.42), vamos usar a métrica para escrever convenientemente alguns termos,

$$\begin{aligned}
\mathcal{L}_{LV} = \xi h^{\mu\nu} \Bigg\{ & p^2 \left(b_\mu b_\nu \eta_{\alpha\beta} + b_\mu b_\beta \eta_{\alpha\nu}\right) - \frac{1}{2} (b \cdot p)^2 \left(\eta_{\mu\alpha} \eta_{\nu\beta} - \eta_{\mu\nu} \eta_{\alpha\beta}\right) \\
& - \left[b_\mu b_\nu p_\alpha p_\beta + \frac{1}{4} (b_\mu p_\nu + b_\nu p_\mu)(b_\alpha p_\beta + b_\beta p_\alpha) \right] \Bigg\} h^{\alpha\beta} \\
& + \frac{4\xi^2}{\kappa^2} h^{\mu\nu} \Bigg[-2p^2 b_\mu b_\nu \eta_{\alpha\beta} - 2b^2 p_\mu p_\nu \eta_{\alpha\beta} + 2(b \cdot p)(b_\mu p_\nu + b_\nu p_\mu) \eta_{\alpha\beta} - \frac{p^2 p_\mu p_\nu \eta_{\alpha\beta}}{2\lambda} \\
& + 2b_\mu b_\nu p_\alpha p_\beta - \frac{1}{4} (b_\mu p_\nu + b_\nu p_\mu)(b_\alpha p_\beta + b_\beta p_\alpha) + \frac{b^2 p_\mu p_\nu p_\alpha p_\beta}{p^2} - \frac{(b \cdot p) p_\mu p_\nu}{p^2} (b_\alpha p_\beta + b_\beta p_\alpha) + \frac{p_\mu p_\nu p_\alpha p_\beta}{4\lambda} \\
& + \left(b^2 p^2 - (b \cdot p)^2 + \frac{p^4}{4\lambda} \right) \eta_{\mu\nu} \eta_{\alpha\beta} + p^2 b_\mu b_\alpha \eta_{\nu\beta} - (b \cdot p)(b_\mu p_\alpha + b_\alpha p_\mu) \eta_{\nu\beta} + \frac{(b \cdot p)^2 p_\mu p_\alpha \eta_{\nu\beta}}{p^2} \Bigg] h^{\alpha\beta} .
\end{aligned} \qquad (4.1.43)$$

A seguir faremos a simetrização dessa lagrangiana, no intuito de encontrar os operadores que surgem da contrbuição dos termos de violação de simetria de Lorentz.

4.1.2 Simetrização da lagrangiana $\mathcal{L}_{LV}$

A extensão da lagrangiana obtida anuncia a dificuldade do próximo passo, que é a sua simetrização. Os termos a serem simetrizados são:

$$b_\mu b_\nu \eta_{\alpha\beta} = \frac{1}{2} \left(b_\mu b_\nu \eta_{\alpha\beta} + b_\alpha b_\beta \eta_{\mu\nu}\right) = \frac{1}{2} \left(\sqrt{3} \tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda - b)}_{\mu\nu,\alpha\beta} \right) , \qquad (4.1.44)$$

$$b_\mu b_\beta \eta_{\alpha\nu} = \frac{1}{4} \left(\eta_{\alpha\nu} b_\mu b_\beta + \eta_{\beta\nu} b_\mu b_\alpha + \eta_{\mu\beta} b_\alpha b_\nu + \eta_{\mu\alpha} b_\beta b_\nu\right) = \frac{1}{4} \left(2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda - a)}_{\mu\nu,\alpha\beta} \right) , \qquad (4.1.45)$$

$$b_\mu b_\nu p_\alpha p_\beta = \frac{1}{2} \left(b_\mu b_\nu p_\alpha p_\beta + b_\alpha b_\beta p_\mu p_\nu\right) = \frac{p^2}{2} \tilde{\Pi}^{(\omega\Lambda - b)}_{\mu\nu,\alpha\beta} ,$$

$$
\begin{aligned}
\left(b_\mu p_\nu + b_\nu p_\mu\right)\left(b_\alpha p_\beta + b_\beta p_\alpha\right) &= b_\mu b_\nu p_\alpha p_\beta + b_\mu b_\alpha p_\nu p_\beta + b_\mu b_\beta p_\nu p_\alpha + b_\nu b_\alpha p_\mu p_\beta + b_\nu b_\beta p_\mu p_\alpha, \\
&= p^2 \tilde{\Pi}^{(\omega\Lambda - a)}_{\mu\nu,\alpha\beta}. \qquad (4.1.46)
\end{aligned}
$$

Este último não foi exatamente simetrizado, mas apenas reescrito. Continuando,

$$
\left(b_\mu p_\nu + b_\nu p_\mu\right)\eta_{\alpha\beta} = \tilde{\Sigma}_{\mu\nu}\eta_{\alpha\beta} = \frac{1}{2}\left(\tilde{\Sigma}_{\mu\nu}\eta_{\alpha\beta} + \tilde{\Sigma}_{\alpha\beta}\eta_{\mu\nu}\right) = \frac{1}{2}\left(\sqrt{3}\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta}\right). \qquad (4.1.47)
$$

$$
p_\mu p_\nu \left(b_\alpha p_\beta + b_\beta p_\alpha\right) = p^2 \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} = \frac{p^2}{2}\left(\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \omega_{\alpha\beta}\tilde{\Sigma}_{\mu\nu}\right) = \frac{p^2}{2}\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta}. \qquad (4.1.48)
$$

$$
b_\mu b_\alpha \eta_{\beta\nu} = \frac{1}{4}\left(\eta_{\alpha\nu} b_\mu b_\beta + \eta_{\beta\nu} b_\mu b_\alpha + \eta_{\mu\beta} b_\alpha b_\nu + \eta_{\mu\alpha} b_\beta b_\nu\right) = \frac{1}{4}\left(2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right), \qquad (4.1.49)
$$

$$
\left(b_\mu p_\alpha + b_\alpha p_\mu\right)\eta_{\nu\beta} = \tilde{\Sigma}_{\mu\alpha}\eta_{\nu\beta} = \frac{1}{4}\left(\eta_{\alpha\nu}\tilde{\Sigma}_{\mu\beta} + \eta_{\beta\nu}\tilde{\Sigma}_{\mu\alpha} + \eta_{\mu\beta}\tilde{\Sigma}_{\alpha\nu} + \eta_{\mu\alpha}\tilde{\Sigma}_{\beta\nu}\right). \qquad (4.1.50)
$$

Neste momento, cumpre observar que:

$$
2\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} = 2\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2\omega_{\alpha\beta}\tilde{\Sigma}_{\mu\nu} = 2\omega_{\mu\nu}\left(b_\alpha p_\beta + b_\beta p_\alpha\right) + 2\omega_{\alpha\beta}\left(b_\mu p_\nu + b_\nu p_\mu\right).
$$

Permutando os índices de ω e $\tilde{\Sigma}$ convenientemente em todas as parcelas, obtemos:

$$
2\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} = \omega_{\alpha\nu}\tilde{\Sigma}_{\mu\beta} + \omega_{\beta\nu}\tilde{\Sigma}_{\mu\alpha} + \omega_{\mu\beta}\tilde{\Sigma}_{\alpha\nu} + \omega_{\mu\alpha}\tilde{\Sigma}_{\beta\nu}. \qquad (4.1.51)
$$

Assim,

$$
\left(b_\mu p_\alpha + b_\alpha p_\mu\right)\eta_{\nu\beta} = \frac{1}{2}\left(\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta}\right). \qquad (4.1.52)
$$

Onde definimos os projetores que surgem na simetrização da lagrangiana $\mathcal{L}_{LV}$,

$$
\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda} &= \frac{\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}}{2}, && (4.1.53)\\
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} &= \frac{\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}}{2}, && (4.1.54)\\
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right), && (4.1.55)\\
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda} + \theta_{\kappa\lambda}\Lambda_{\mu\nu}\right), && (4.1.56)\\
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} &= \omega_{\mu\kappa}\Lambda_{\nu\lambda} + \omega_{\mu\lambda}\Lambda_{\nu\kappa} + \omega_{\nu\kappa}\Lambda_{\mu\lambda} + \omega_{\nu\lambda}\Lambda_{\mu\kappa}, && (4.1.57)\\
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} &= \omega_{\mu\nu}\Lambda_{\kappa\lambda} + \omega_{\kappa\lambda}\Lambda_{\mu\nu}, && (4.1.58)\\
\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda} &= \omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}, && (4.1.59)
\end{aligned}
$$

onde $\tilde{\Sigma}_{\mu\nu} = b_\mu p_\nu + b_\nu p_\mu$ e $\Lambda_{\mu\nu} = b_\mu b_\nu$. Reescrevemos a lagrangiana (4.1.42) no seguinte formato:

$$
\begin{aligned}
\mathcal{L}_{LV} = \xi h^{\mu\nu} \Bigg\{ & p^2 \left[\frac{1}{2} \left(\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta} \right) + \frac{1}{4} \left(2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \right) \right] \\
& - \frac{1}{2} (b\cdot p)^2 \left[P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(1)}_{\mu\nu,\alpha\beta} + P^{(2)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \left(3P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \sqrt{3} P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} \right) \right] \\
& - p^2 \left(\frac{1}{2} \tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta} + \frac{1}{4} \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \right) \Bigg\} h^{\alpha\beta} + \frac{4\xi^2}{\kappa^2} h^{\mu\nu} \left[-p^2 \left(\sqrt{3} \tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta} \right) \right. \\
& - b^2 p^2 \left(2P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \sqrt{3} P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} \right) + (b\cdot p) \left(\sqrt{3} \tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} \right) - \frac{p^4}{4\lambda} \left(2P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \sqrt{3} P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} \right) \\
& + p^2 \tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta} - \frac{1}{4} p^2 \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + b^2 p^2 P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \frac{1}{2} (b\cdot p) \tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} + \frac{p^4}{4\lambda} P^{(0-\omega)}_{\mu\nu,\alpha\beta} \\
& + \left(b^2 p^2 - (b\cdot p)^2 + \frac{p^4}{4\lambda} \right) \left(3P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \sqrt{3} P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} \right) + \frac{p^2}{4} \left(2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \right) \\
& \left. - \frac{1}{2} (b\cdot p) \left(\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} \right) + (b\cdot p)^2 \left(\frac{1}{2} P^{(1)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta} \right) \right] h^{\alpha\beta}. \qquad (4.1.60)
\end{aligned}
$$

Vamos juntar os termos semelhantes e simplificar para obter o resultado:

$$
\begin{aligned}
\mathcal{L}_{LV} = \frac{1}{2} h^{\mu\nu} \Bigg\{ & \left[-\xi (b\cdot p)^2 + \frac{4\xi^2}{\kappa^2} (b\cdot p)^2 \right] P^{(1)}_{\mu\nu,\alpha\beta} - \xi (b\cdot p)^2 P^{(2)}_{\mu\nu,\alpha\beta} \\
& + \left[2\xi (b\cdot p)^2 + \frac{24\xi^2}{\kappa^2} \left(b^2 p^2 - (b\cdot p)^2 + \frac{p^4}{4\lambda} \right) \right] P^{(0-\theta)}_{\mu\nu,\alpha\beta} \\
& + \left[\sqrt{3}\xi (b\cdot p)^2 - \frac{8\sqrt{3}\xi^2}{\kappa^2} (b\cdot p)^2 \right] P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} - \frac{4\xi^2}{\kappa^2} (b\cdot p) \tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + \left(\xi p^2 + \frac{4\xi^2}{\kappa^2} p^2 \right) \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} \\
& + \frac{8\sqrt{3}\xi^2}{\kappa^2} (b\cdot p) \tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} + \left(\sqrt{3}\xi p^2 - \frac{8\sqrt{3}\xi^2}{\kappa^2} p^2 \right) \tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} \Bigg\} h^{\alpha\beta}. \qquad (4.1.61)
\end{aligned}
$$

Aqui faremos um comentário importante: note que os elementos que compõem a base que "quadra" a lagrangiana $\mathcal{L}_{LV}$ é composta pelos projetores $\left[P^{(1)}_{\mu\nu,\alpha\beta}, P^{(2)}_{\mu\nu,\alpha\beta}, P^{(0-\theta)}_{\mu\nu,\alpha\beta}, P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}, \tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta}, \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}, \tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} \text{ e } \tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} \right]$. No entanto, a fim de compor uma álgebra fechada que permita calcular o propagador do gráviton de Einstein-Hilbert modificado por termos de violação de Lorentz, é necessário acrescentar mais projetores, de modo que o conjunto total de operadores é exibido a seguir :

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda} = \frac{\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}}{2}, \tag{4.1.62}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} = \frac{\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}}{2}, \tag{4.1.63}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right), \tag{4.1.64}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda} + \theta_{\kappa\lambda}\Lambda_{\mu\nu}\right), \tag{4.1.65}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda} = \Lambda_{\mu\nu}\Lambda_{\kappa\lambda}, \tag{4.1.66}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\kappa}\Lambda_{\nu\lambda} + \omega_{\mu\lambda}\Lambda_{\nu\kappa} + \omega_{\nu\kappa}\Lambda_{\mu\lambda} + \omega_{\nu\lambda}\Lambda_{\mu\kappa}, \tag{4.1.67}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\nu}\Lambda_{\kappa\lambda} + \omega_{\kappa\lambda}\Lambda_{\mu\nu}, \tag{4.1.68}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda} = \omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}, \tag{4.1.69}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda} = \Lambda_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}. \tag{4.1.70}$$

Em outras palavras, ao efetuarmos a contração entre os projetores $\Big[P^{(1)}_{\mu\nu,\alpha\beta},\ P^{(2)}_{\mu\nu,\alpha\beta},\ P^{(0-\theta)}_{\mu\nu,\alpha\beta},\ P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta},\ \tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta},\ \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta},\ \tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta}$ e $\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta}\Big]$ vão surgindo termos que não se identificam com nenhum destes, o procedimento de fechamento é similar ao realizado para o caso da gravitação de Chern-Simons, e o conjunto de operadores que deve ser introduzido está representado eqs. (4.1.62)-(4.1.70).

A lagrangiana total, composta pelas contribuições de Einstein-Hilbert (sem o termo de gauge-fixing) e violação de Lorentz é dada por:

$$\mathcal{L} = \mathcal{L}_{EH} + \mathcal{L}_{LV}. \tag{4.1.71}$$

Lembrando que $\mathcal{L}_{EH}$ também pode ser escrita como,

$$\mathcal{L}_{EH} = \frac{1}{2}h^{\mu\nu}\left(-\eta_{\mu\nu}\eta_{\alpha\beta}p^2 + 2\eta_{\mu\nu}p_\alpha p_\beta + \eta_{\mu\alpha}\eta_{\nu\beta}p^2 - 2\eta_{\nu\alpha}p_\beta p_\mu\right)h^{\alpha\beta}, \tag{4.1.72}$$

obtemos para a eq. (4.1.71) o seguinte resultado:

$$\mathcal{L} = -\frac{1}{2}h^{\mu\nu}\mathcal{O}_{\mu\nu,\alpha\beta}h^{\alpha\beta}, \tag{4.1.73}$$

onde

$$\begin{aligned}\mathcal{O}_{\mu\nu,\alpha\beta} = {} & a_1P^{(1)}_{\mu\nu,\alpha\beta} + a_2P^{(2)}_{\mu\nu,\alpha\beta} + a_3P^{(0-\theta)}_{\mu\nu,\alpha\beta} + a_4P^{(0-\omega)}_{\mu\nu,\alpha\beta} + a_5P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} \\ & + a_6\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + a_7\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + a_8\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} + a_9\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta},\end{aligned} \tag{4.1.74}$$

e os coeficientes a_i são dados por:

$$a_1 = \xi\,(b\cdot p)^2 - \frac{4\xi^2}{\kappa^2}(b\cdot p)^2, \qquad a_2 = p^2 + \xi\,(b\cdot p)^2, \tag{4.1.75}$$

$$a_3 = -\xi\,(b\cdot p)^2 - \frac{24\xi^2}{\kappa^2}\left(b^2p^2 - (b\cdot p)^2 + \frac{p^4}{4\lambda}\right) - 2p^2 \tag{4.1.76}$$

$$a_4 = \frac{8\sqrt{3}\xi^2}{\kappa^2}(b\cdot p)^2 - \sqrt{3}\xi\,(b\cdot p)^2, \qquad a_5 = \frac{4\xi^2}{\kappa^2}(b\cdot p), \tag{4.1.77}$$

$$a_6 = -\xi p^2 - \frac{4\xi^2}{\kappa^2}p^2, \qquad a_7 = -\frac{8\sqrt{3}\xi^2}{\kappa^2}(b\cdot p), \qquad a_8 = \frac{8\sqrt{3}\xi^2}{\kappa^2}p^2 - \sqrt{3}\xi p^2, \tag{4.1.78}$$

resultado inteiramente de acordo com [22].

4.2 Cálculo do propagador do gráviton com violação da simetria de Lorentz

O propagador do gráviton no cenário com violação da simetria de Lorentz tem como estrutura o operador $\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}$, que satisfaz a relação

$$\mathcal{O}_{\mu\nu,\kappa\lambda}\left(\mathcal{O}^{\kappa\lambda}{}_{,\alpha\beta}\right)^{-1}=\mathcal{I}_{\mu\nu,\alpha\beta}, \tag{4.2.1}$$

onde $\mathcal{I}_{\mu\nu,\alpha\beta}$ é o operador identidade dado na eq. (2.2.30). A forma mais geral proposta para o operador $\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}$ é uma combinação linear de todos os projetores da base de Barnes-Rivers estendida,

$$\begin{aligned}\mathcal{O}^{-1}_{\mu\nu,\alpha\beta}=b_1P^{(1)}_{\mu\nu,\alpha\beta}+b_2P^{(2)}_{\mu\nu,\alpha\beta}+b_3P^{(0-\theta)}_{\mu\nu,\alpha\beta}+b_4P^{(0-\omega)}_{\mu\nu,\alpha\beta}+b_5P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}\\+b_6\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta}+b_7\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+b_8\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta}+b_9\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta}\\+b_{10}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}+b_{11}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}+b_{12}\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta}+b_{13}\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta}+b_{14}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\alpha\beta}.\end{aligned} \tag{4.2.2}$$

O desafio maior nesse processo é obter corretamente as contrações que envolvem todos os operadores da base estendida. Um outro detalhe que precisa ser mencionado é o seguinte: vimos que as contrações que envolvem $P^{(0-\theta)}_{\mu\nu,\alpha\beta}$, $P^{(0-\omega)}_{\mu\nu,\alpha\beta}$, e $P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}$, quando não nulas são não comutativas, ou seja,

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}\neq P^{(0-\theta)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta}, \tag{4.2.3}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}\neq P^{(0-\omega)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta}. \tag{4.2.4}$$

Dentro da base de Barnes-Rivers essas são as únicas contrações entre diferentes projetores que possuem resultado diferente de zero (vide tabela 2.2). Esse fato chama atenção para a importância de observar a maneira como os índices dos operadores são contraídos. É importantíssimo frisar esse detalhe porque um estudante desatento pode pensar que basta calcular uma contração que a outra terá o mesmo resultado, o que além de não ser verdade, acarreta em um erro grave. O sistema montado com os resultados incorretos das contrações seria trivial, caso exista uma solução não-trivial (possibilidade baixíssima devido à complexidade do sistema obtido), estaria igualmente incorreta.

Uma outra informação relevante é que não é necessário calcular duas vezes a contração entre dois projetores distintos, mas apenas trocar os índices dos termos não-simetrizados é suficiente para a obtenção do resultado. Vamos exemplificar isso na primeira contração, o restante foi obtido apenas pela troca dos índices. Iniciamos pelas contração com $P^{(0-\theta)}_{\mu\nu,\alpha\beta}$:

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}&=\frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}+\theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}+\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta},\\&=\frac{1}{6}\left(\theta^{\lambda}{}_{\mu}\tilde{\Sigma}_{\nu\lambda}+\theta^{\kappa}{}_{\mu}\tilde{\Sigma}_{\nu\kappa}+\theta^{\lambda}{}_{\nu}\tilde{\Sigma}_{\mu\lambda}+\theta^{\kappa}{}_{\nu}\tilde{\Sigma}_{\mu\kappa}\right)\theta_{\alpha\beta}.\end{aligned} \tag{4.2.5}$$

Notando que,

$$\tilde{\Sigma}_{\nu\lambda}\theta^{\lambda}{}_{\beta}=\left(b_\nu p_\lambda+b_\lambda p_\nu\right)\theta^{\lambda}{}_{\beta}=b_\beta p_\nu-\left(b\cdot p\right)\omega_{\nu\beta}, \tag{4.2.6}$$

decorre:

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}&=\frac{1}{6}\left[2\left(b_\mu p_\nu-\left(b\cdot p\right)\omega_{\mu\nu}\right)+2\left(b_\nu p_\mu-\left(b\cdot p\right)\omega_{\mu\nu}\right)\right]\theta_{\alpha\beta},\\\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}&=\frac{1}{3}\left[\theta_{\alpha\beta}b_\mu p_\nu+\theta_{\alpha\beta}b_\nu p_\mu-\left(b\cdot p\right)\left(\omega_{\mu\nu}\theta_{\alpha\beta}+\omega_{\mu\nu}\theta_{\alpha\beta}\right)\right],\\\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}&=\frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}-\frac{2}{3}\left(b\cdot p\right)\omega_{\mu\nu}\theta_{\alpha\beta}.\end{aligned} \tag{4.2.7}$$

Invertendo a ordem, temos

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda} \times \frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\tilde{\Sigma}^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\tilde{\Sigma}^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\tilde{\Sigma}^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\tilde{\Sigma}^{\kappa}{}_{\alpha}\right), \tag{4.2.8}$$

$$= \frac{1}{3}\theta_{\mu\nu}\theta_{\kappa\lambda}\left(\theta^{\kappa}{}_{\alpha}\tilde{\Sigma}^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\beta}\tilde{\Sigma}^{\kappa}{}_{\alpha}\right) = \frac{1}{3}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2}{3}(b\cdot p)\,\theta_{\mu\nu}\omega_{\alpha\beta}. \tag{4.2.9}$$

Observe que o resultado (4.2.8) poderia ser obtido a partir da eq. (4.2.7) através da troca dos índices de θ e $\tilde{\Sigma}$. A razão disto é bem simples: ao inverter-se a ordem não mudam-se os projetores, mas apenas os índices contraídos e os pares de índices livres. Por razões estéticas, o restante do cálculo das contrações, bem como todos os outros cálculos mais extensos que envolvem o propagador, estão presentes nos apêndices E e F. Abaixo estão listadas apenas as contrações (não-nulas) dos operadores de Barnes-Rivers com os novos projetores de spin que carregam o campo bumblebee:

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} - \frac{2}{3}(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.10}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2}{3}(b\cdot p)\,\theta_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.11}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.12}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{3}\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.13}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.14}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.15}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.16}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.17}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.18}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.19}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.20}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.21}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.22}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.23}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\left(b\cdot p\right)P^{(1)}_{\mu\nu,\alpha\beta}+\Pi^{(1-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.24}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta}=\left(b\cdot p\right)P^{(1)}_{\mu\nu,\alpha\beta}+\Pi^{(1-b)}_{\mu\nu,\alpha\beta}, \tag{4.2.25}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{p^2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.26}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{p^2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta}, \tag{4.2.27}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)\theta_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.28}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)\omega_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.29}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.30}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}-2\frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.31}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\frac{(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.32}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-2\frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.33}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-2\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.34}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-2\frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.35}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{2\left(b\cdot p\right)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.36}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}-\frac{2\left(b\cdot p\right)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.37}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\left(b\cdot p\right)P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.38}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}-2\left(b\cdot p\right)P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.39}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta}+\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\frac{2\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.40}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-2\left(b\cdot p\right)\omega_{\mu\nu}\Lambda_{\alpha\beta}+\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\frac{2\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.41}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta}=\Pi^{(1-b)}_{\mu\nu,\alpha\beta}-\left(b\cdot p\right)P^{(1)}_{\mu\nu,\alpha\beta}-\frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\frac{2}{3}\left(b\cdot p\right)\omega_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.42}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta}=\Pi^{(1-a)}_{\mu\nu,\alpha\beta}-\left(b\cdot p\right)P^{(1)}_{\mu\nu,\alpha\beta}-\frac{1}{3}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{2}{3}\left(b\cdot p\right)\theta_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.43}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}-\frac{(b\cdot p)}{p^2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta}-\frac{2}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{(b\cdot p)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.44}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}-\frac{(b\cdot p)}{p^2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta}-\frac{2}{3}\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{(b\cdot p)}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\tag{4.2.45}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}-\frac{(b\cdot p)}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta},\tag{4.2.46}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}-\frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta},\tag{4.2.47}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{3p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta},\tag{4.2.48}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta}=\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}+\frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta},\tag{4.2.49}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta}=\omega_{\mu\nu}\Lambda_{\alpha\beta}-\frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta},\tag{4.2.50}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta}=\Lambda_{\mu\nu}\omega_{\alpha\beta}-\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}+\frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.51}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta}=\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}+\frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta},\tag{4.2.52}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}+\frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\tag{4.2.53}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}=\frac{2}{\sqrt{3}}\,(b\cdot p)\,\theta_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.54}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\frac{2}{\sqrt{3}}\,(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta},\tag{4.2.55}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.56}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta},\tag{4.2.57}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.58}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta},\tag{4.2.59}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}=\frac{2\,(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.60}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}{}_{,\alpha\beta}=\frac{2\,(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\tag{4.2.61}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}+\Lambda_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.62}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}+\omega_{\mu\nu}\Lambda_{\alpha\beta},\tag{4.2.63}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.64}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.65}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,\Lambda_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.66}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.67}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - \frac{2}{\sqrt{3}}\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.68}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2}{\sqrt{3}}\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.69}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.70}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.71}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.72}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.73}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta} + \Lambda_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.74}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta} + \omega_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.75}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.76}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.77}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{2\,(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.78}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}{}_{,\alpha\beta} = \frac{2\,(b\cdot p)}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.79}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.80}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.81}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\,(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.82}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\,(b\cdot p)\,\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.83}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(\ 0-\theta\omega)\ \kappa\lambda}{}_{,\alpha\beta}=\frac{2}{\sqrt{3}}\left(b\cdot p\right)\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{\left(b\cdot p\right)^{2}}{\sqrt{3}p^{2}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\frac{b^{2}p^{2}-\left(b\cdot p\right)^{2}}{\sqrt{3}p^{2}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.84}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\frac{2}{\sqrt{3}}\left(b\cdot p\right)\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{\left(b\cdot p\right)^{2}}{\sqrt{3}p^{2}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{b^{2}p^{2}-\left(b\cdot p\right)^{2}}{\sqrt{3}p^{2}}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\tag{4.2.85}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=&\frac{1}{2}p^{2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta}+2\left(b\cdot p\right)^{2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}\\&+p^{2}\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+\left(b\cdot p\right)\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta}+b^{2}p^{2}P^{(1)}_{\mu\nu,\alpha\beta},\end{aligned}\tag{4.2.86}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta}=&\left(b\cdot p\right)\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+b^{2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta}+\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\\&-\frac{1}{2}\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-2\left(b\cdot p\right)\omega_{\mu\nu}\Lambda_{\alpha\beta}+\frac{\left(b\cdot p\right)^{2}}{p^{2}}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\end{aligned}\tag{4.2.87}$$

$$\begin{aligned}\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=&\left(b\cdot p\right)\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+b^{2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta}+\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\\&-\frac{1}{2}\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta}+\frac{\left(b\cdot p\right)^{2}}{p^{2}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\end{aligned}\tag{4.2.88}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta}=\frac{1}{\sqrt{3}}p^{2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)^{2}\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{2}{\sqrt{3}}p^{2}\Lambda_{\mu\nu}\theta_{\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\tag{4.2.89}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=\frac{1}{\sqrt{3}}p^{2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)^{2}\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{2}{\sqrt{3}}p^{2}\theta_{\mu\nu}\Lambda_{\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.90}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta}=&\frac{b^{2}p^{2}-\left(b\cdot p\right)^{2}}{\sqrt{3}p^{2}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\frac{2}{\sqrt{3}}\left(b\cdot p\right)\Lambda_{\mu\nu}\theta_{\alpha\beta}\\&-\frac{2}{\sqrt{3}}\left(b\cdot p\right)b^{2}\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)\omega_{\mu\nu}\Lambda_{\alpha\beta},\end{aligned}\tag{4.2.91}$$

$$\begin{aligned}\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=&\frac{b^{2}p^{2}-\left(b\cdot p\right)^{2}}{\sqrt{3}p^{2}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{2}{\sqrt{3}}\left(b\cdot p\right)\theta_{\mu\nu}\Lambda_{\alpha\beta}\\&-\frac{2}{\sqrt{3}}\left(b\cdot p\right)b^{2}\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{2}{\sqrt{3}}\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta},\end{aligned}\tag{4.2.92}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta}=\frac{b^{2}p^{2}-\left(b\cdot p\right)^{2}}{p^{2}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}+2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-2\left(b\cdot p\right)b^{2}\omega_{\mu\nu}\Lambda_{\alpha\beta},\tag{4.2.93}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=\frac{b^{2}p^{2}-\left(b\cdot p\right)^{2}}{p^{2}}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-2\left(b\cdot p\right)b^{2}\Lambda_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.94}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta}=2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{2\left(b\cdot p\right)^{2}}{p^{2}}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\tag{4.2.95}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=2\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-\frac{2\left(b\cdot p\right)^{2}}{p^{2}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\tag{4.2.96}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} = \frac{b^2p^2-(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + 2(b\cdot p)\Lambda_{\mu\nu}\omega_{\alpha\beta} - 2(b\cdot p)b^2P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.97}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{b^2p^2-(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2(b\cdot p)\omega_{\mu\nu}\Lambda_{\alpha\beta} - 2(b\cdot p)b^2P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.98}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}_{,\alpha\beta} = 2p^2\Lambda_{\mu\nu}\omega_{\alpha\beta} - 2(b\cdot p)^2P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.99}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = 2p^2\omega_{\mu\nu}\Lambda_{\alpha\beta} - 2(b\cdot p)^2P^{(0-\omega)}_{\mu\nu,\alpha\beta}, \tag{4.2.100}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} &= 2p^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2(b\cdot p)^2\omega_{\mu\nu}\Lambda_{\alpha\beta}\\ &\quad + \left[b^2p^2-(b\cdot p)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + 2(b\cdot p)\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2(b\cdot p)b^2\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\end{aligned} \tag{4.2.101}$$

$$\begin{aligned}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} &= 2p^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2(b\cdot p)^2\Lambda_{\mu\nu}\omega_{\alpha\beta}\\ &\quad + \left[b^2p^2-(b\cdot p)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + 2(b\cdot p)\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - 2(b\cdot p)b^2\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\end{aligned} \tag{4.2.102}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = b^2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + 2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\alpha\beta} + \frac{(b\cdot p)^2}{2p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.103}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{1}{\sqrt{3}}(b\cdot p)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.104}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{1}{\sqrt{3}}(b\cdot p)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.105}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}b^2\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)b^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.106}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)b^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{\sqrt{3}}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.107}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} = 2b^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.108}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2b^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)b^2}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.109}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = \frac{2(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.110}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{2(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.111}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} = 2b^2\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{(b\cdot p)b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.112}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2b^2\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)b^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.113}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.114}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.115}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} + 2b^2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \left(b\cdot p\right)b^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.116}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{\left(b\cdot p\right)^2}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2b^2\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \left(b\cdot p\right)b^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.117}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = 2\left[b^2p^2 + \left(b\cdot p\right)^2\right]P^{(0-\theta)}_{\mu\nu,\alpha\beta} + p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \tag{4.2.118}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)b^2P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.119}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)b^2P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.120}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}, \tag{4.2.121}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.122}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = \frac{2\left[b^2p^2 + \left(b\cdot p\right)^2\right]}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.123}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2\left[b^2p^2 + \left(b\cdot p\right)^2\right]}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.124}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{2}{\sqrt{3}}\left(b\cdot p\right)\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \tag{4.2.125}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\left(b\cdot p\right)\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.126}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left[b^2p^2 + \left(b\cdot p\right)^2\right]\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{2}{\sqrt{3}}\left(b\cdot p\right)\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \tag{4.2.127}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left[b^2p^2 + \left(b\cdot p\right)^2\right]\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\left(b\cdot p\right)\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \tag{4.2.128}$$

$$\begin{aligned}\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} = {}& \frac{2}{\sqrt{3}}\left[b^2p^2 + \left(b\cdot p\right)^2\right]\theta_{\mu\nu}\Lambda_{\alpha\beta} \\ & + \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{1}{\sqrt{3}}\left[b^2p^2 - \left(b\cdot p\right)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta},\end{aligned} \tag{4.2.129}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left[b^2p^2 + (b\cdot p)^2\right]\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\left[b^2p^2 - (b\cdot p)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}, \quad (4.2.130)$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = b^4P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}, \quad (4.2.131)$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b^4\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}, \quad (4.2.132)$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b^4\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}, \quad (4.2.133)$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = \frac{2\,(b\cdot p)\,b^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \quad (4.2.134)$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2\,(b\cdot p)\,b^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \quad (4.2.135)$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b^4\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}, \quad (4.2.136)$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b^4\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}, \quad (4.2.137)$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\ \kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \quad (4.2.138)$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \quad (4.2.139)$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\ \kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{1}{\sqrt{3}}b^4\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \quad (4.2.140)$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}b^4\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}. \quad (4.2.141)$$

Após calcularmos todos os produtos de contração tensorial entre os projetores da base estendida, é preciso substituí-los na expressão (4.2.1) e coletar os termos semelhantes, formando um sistema de equações (veja o apêndice F), por meio de comparação com os projetores que constituem o operador identidade dado na eq. (2.2.30). A solução do sistema obtido fornece:

$$
\begin{aligned}
b_1 &= \frac{N_1}{\kappa^2\xi^2\left(b\cdot p\right)^2\boxdot\boxplus}, \quad b_2=\frac{1}{\boxplus}, \quad b_3=-\frac{1}{2\boxplus},\\
b_4 &= \frac{N_4}{2\lambda\kappa^2\xi^2\left(b\cdot p\right)^4\boxdot^2\boxplus}, \quad b_5=\frac{N_5}{2\xi\left(b\cdot p\right)^2\boxdot\boxplus},\\
b_6 &= \frac{p^2}{\xi\left(b\cdot p\right)\boxdot\boxplus}, \quad b_7=\frac{p^2}{\boxdot\boxplus}, \quad b_8=\frac{N_8}{4\xi\left(b\cdot p\right)\boxdot\boxplus},\\
b_9 &= -\frac{\sqrt{3}p^2}{2\boxdot\boxplus}, \quad b_{10}=\frac{p^4}{2\boxdot^2\boxplus}, \quad b_{11}=\frac{N_{11}}{8\xi^2\left(b\cdot p\right)^2\boxdot^2\boxplus},\\
b_{12} &= \frac{N_{12}}{2\xi\left(b\cdot p\right)^2\boxdot^2\boxplus}, \quad b_{13}=\frac{N_{13}}{4\kappa^2\xi^2\left(b\cdot p\right)^3\boxdot^2\boxplus},\\
b_{14} &= \frac{N_{14}}{4\xi\left(b\cdot p\right)\boxdot^2\boxplus},
\end{aligned}
\tag{4.2.142}
$$

onde os símbolos $\boxdot$ e $\boxplus$ são funções do momento (p^μ) e do campo bumblebee (b^μ), possuindo o seguinte formato:

$$\boxplus = p^2+\xi\left(b\cdot p\right)^2, \tag{4.2.143}$$

$$\boxdot = \left(b\cdot p\right)^2-b^2p^2. \tag{4.2.144}$$

Já os coeficientes representados por N_i são:

$$N_1=\xi\left(4\xi+\kappa^2\right)\boxdot\boxplus+\kappa^2p^4, \tag{4.2.145}$$

$$
\begin{aligned}
N_4 = {}& \xi^2\boxdot^2\left[p^2F_1\left(p\right)+\lambda\kappa^2\left(b\cdot p\right)^4\right]+4\lambda\xi\kappa^2p^2\left(b\cdot p\right)^4\boxdot\\
&+\xi^3\left(b\cdot p\right)^2\boxdot^2F_1\left(p\right)-\lambda\kappa^2p^4\left[b^4p^4-4\left(b\cdot p\right)^4+2b^2p^2\left(b\cdot p\right)^2\right],
\end{aligned}
\tag{4.2.146}
$$

$$N_5=\sqrt{3}\left[b^2p^4-\xi\left(b\cdot p\right)^2\boxdot\right], \qquad N_8=\sqrt{3}\left[\xi\left(b\cdot p\right)^2-p^2\right], \tag{4.2.147}$$

$$N_{11}=p^2\left[p^2-\xi\left(b\cdot p\right)^2\right]^2, \qquad N_{12}=p^4\left[2b^2p^2-3\left(b\cdot p\right)^2\right]-2\xi p^2\left(b\cdot p\right)^2\boxdot, \tag{4.2.148}$$

$$N_{13}=F_1\left(p\right)\xi^2\boxdot-16\xi^3\left(b\cdot p\right)^2\boxdot^2+\xi\kappa^2b^2p^4\left[2\left(b\cdot p\right)^2-b^2p^2\right]+\kappa^2p^4\left[2b^2p^2-3\left(b\cdot p\right)^2\right], \tag{4.2.149}$$

$$N_{14}=p^2\left[p^2-\xi\left(b\cdot p\right)^2\right], \tag{4.2.150}$$

sendo $F_1\left(p\right)$ e $F_2\left(p\right)$ dados por:

$$F_1\left(p\right)=16\lambda\left[\left(b\cdot p\right)^2+b^2p^2\right]+p^4, \tag{4.2.151}$$

$$F_2\left(p\right)=\kappa^2\left(b\cdot p\right)^2\left[\left(b\cdot p\right)^2+b^2p^2\right]+16p^2\boxdot. \tag{4.2.152}$$

Com isso, o propagador de Feynmann para o gráviton no cenário com violação da simetria de Lorentz é

$$
\begin{aligned}
\Delta_{\mu\nu,\alpha\beta} = \frac{i}{\boxplus}\Bigg\{ & \frac{N_1}{\kappa^2\xi^2\left(b\cdot p\right)^2\boxdot}P^{(1)}_{\mu\nu,\alpha\beta}+P^{(2)}_{\mu\nu,\alpha\beta}-\frac{1}{2}P^{(0-\theta)}_{\mu\nu,\alpha\beta}+\frac{N_4}{2\lambda\kappa^2\xi^2\left(b\cdot p\right)^4\boxdot^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}\\
&+\frac{N_5}{2\xi\left(b\cdot p\right)^2\boxdot}P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}+\frac{p^2}{\xi\left(b\cdot p\right)\boxdot}\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta}+\frac{p^2}{\boxdot}\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+\frac{N_8}{4\xi\left(b\cdot p\right)\boxdot}\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta}\\
&-\frac{\sqrt{3}p^2}{2\boxdot}\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta}+\frac{p^4}{2\boxdot^2}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}+\frac{N_{11}}{8\xi^2\left(b\cdot p\right)^2\boxdot^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}+\frac{N_{12}}{2\xi\left(b\cdot p\right)^2\boxdot^2}\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta}\\
&+\frac{N_{13}}{4\kappa^2\xi^2\left(b\cdot p\right)^3\boxdot^2}\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta}+\frac{N_{14}}{4\xi\left(b\cdot p\right)\boxdot^2}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\alpha\beta}\Bigg\}.
\end{aligned}
\tag{4.2.153}
$$

A partir do propagador $\Delta_{\mu\nu,\alpha\beta}$ extraímos as seguintes relações de dispersão

$$\boxplus = p^2 + \xi \left(b \cdot p\right)^2 = 0, \tag{4.2.154}$$

$$\boxdot = \left(b \cdot p\right)^2 - b^2 p^2 = 0. \tag{4.2.155}$$

Vamos analisar cada uma separadamente e verificar se os modos obtidos são causais. Para isso, consideraremos duas diferentes configurações para o campo bumblebee, uma tipo-tempo, $b^\mu = (b_0, \mathbf{0})$, e a outra do tipo-espaço, $b^\mu = (0, \mathbf{b})$.

As raízes de $\boxplus = 0$, para um campo de fundo genérico $b^\mu = (b_0, \mathbf{b})$ são dadas por:

$$p_0 = \frac{\xi b_0 \left|\mathbf{b}\right| \left|\mathbf{p}\right| \cos\theta \pm \left|\mathbf{p}\right| \sqrt{1 + \xi \left(b_0^2 - \left|\mathbf{b}\right|^2 \cos^2\theta\right)}}{1 + \xi b_0^2}, \tag{4.2.156}$$

onde usamos $\mathbf{b} \cdot \mathbf{p} = \left|\mathbf{b}\right| \left|\mathbf{p}\right| \cos\theta$. Note não haver nenhuma restrição explícita quanto à configuração do campo b^μ, ou seja, p_0 possui solução bem definida em ambas configurações: tipo-espaço e tipo-tempo. Iniciamos considerando o caso em que o campo bumblebee é tipo-tempo na eq. (4.2.156), que produz:

$$p_0 = \pm \frac{\left|\mathbf{p}\right| \sqrt{1 + \xi b_0^2}}{1 + \xi b_0^2} = \pm \frac{\left|\mathbf{p}\right|}{\sqrt{1 + \xi b_0^2}}. \tag{4.2.157}$$

A velocidade de grupo associada é dada por

$$u_g = \frac{1}{\sqrt{1 + \xi b_0^2}}, \tag{4.2.158}$$

sendo menor que 1 desde que $\xi > 0$. A velocidade de fase u_f será

$$u_f = \frac{p_0}{\left|\mathbf{p}\right|} = u_g < 1. \tag{4.2.159}$$

Portanto, este modo é causal para uma configuração do tipo-tempo com $\xi > 0$. Vamos examinar o caso de uma configuração tipo-espaço, neste caso a eq. (4.2.156) torna-se:

$$p_0 = \pm \left|\mathbf{p}\right| \sqrt{1 - \xi \left|\mathbf{b}\right|^2 \cos^2\theta}. \tag{4.2.160}$$

Neste caso, a velocidade de grupo tem a forma a seguir:

$$u_g = \sqrt{1 - \xi \left|\mathbf{b}\right|^2 \cos^2\theta}, \tag{4.2.161}$$

que será menor que 1 para $\xi > 0$, e $\xi \left|\mathbf{b}\right|^2 < 1$. Em resumo, o modo de propagação associado ao polo $\boxplus = 0$ é causal nas duas configurações possíveis para b^μ com a mesma restrição sobre a constante ξ ($\xi > 0$) e a restrição adicional $\xi \left|\mathbf{b}\right|^2 < 1$.

Vamos agora olhar para o segundo polo $\boxdot = 0$. A solução geral deste modo é dada pela expressão

$$p_0 = \frac{\left|\mathbf{p}\right|}{\left|\mathbf{b}\right|} \left[b_0 \cos\theta \pm \sqrt{\left(\left|\mathbf{b}\right|^2 - b_0^2\right) \sin^2\theta}\right]. \tag{4.2.162}$$

Note que agora temos uma restrição importante, a raiz quadrada não é definida para números negativos (dentro de $\mathbb{R}$). Portanto, devemos ter

$$\left|\mathbf{b}\right|^2 - b_0^2 > 0, \tag{4.2.163}$$

que representa uma configuração tipo-espaço, ou seja, para o polo $\boxdot = 0$, o campo b^μ não admite uma configuração tipo-tempo. Assim, para $b^\mu = (0, \mathbf{b})$, a eq. (4.2.162) torna-se

$$p_0 = \pm |\mathbf{p}| \sin\theta. \tag{4.2.164}$$

A velocidade de grupo desse modo equivale a

$$u_g = \sin\theta \leq 1, \tag{4.2.165}$$

sendo igual à velocidade de fase u_f. A relação de dispersão representada na eq. (4.2.164), embora esteja de acordo com o princípio da causalidade, reflete um comportamento físico exótico, onde a energia do modo depende crucialmente da direção de propagação, ou melhor, o ângulo formado entre o momento linear do gráviton e o vetor de fundo $\mathbf{b}$, que representa o campo de violação da simetria de Lorentz.

CAPÍTULO 5

Conclusões e perspectivas

Esta dissertação tratou da revisão de conceitos da gravitação de Einstein-Hilbert, com ênfase no cálculo do propagador de Feynman do gráviton na teoria gravitacional usual e em modelos gravitacionais alternativos. Os desenvolvimentos aqui realizados podem ser aplicados em outros cenários ainda não explorados na literatura, o que constitui uma motivação adicional para esse estudo

No capítulo 2, realizamos uma breve descrição do método de projetores tensoriais para o cálculo do propagador em teorias de calibre de spin-1, apenas para familiarizar o leitor com o formalismo que foi empregado em todo este trabalho.

O capítulo 3 destina-se a apresentar a base de Barnes-Rivers como ferramental apropriado para calcular o propagador do gráviton em qualquer teoria gravitacional não-massiva, e cuja lagrangiana seja escrita apenas em termos do tensor de Ricci e suas respectivas contrações. No final, aplicamos o resultado obtido para a gravitação de Einstein-Hilbert e verificamos que a relação de dispersão dessa teoria é compatível com a das ondas gravitacionais planas, com velocidade de propagação igual a c.

No quarto capítulo abordamos dois modelos gravitacionais alternativos, o primeiro consiste em uma versão estendida da teoria gravitacional de Einstein-Hilbert modificada por termos do tipo R^2 e $R_{\mu\nu}R^{\mu\nu}$. O propagador do gráviton obtido nesses nos permite obter dois novos modos massivos de propagação: um possui relação de dispersão em conformidade com a causalidade, porém, o outro modo a viola. Este resultado evidencia a dificuldade de se obter uma teoria gravitacional consistente (sem táquions) e que seja renormalizável, pois um gráviton massivo, por si só, não é necessariamente problemático, podendo tornar a gravitação renormalizável sem comprometer a unitariedade [59]. Ainda no capítulo 4, consideramos a gravitação de Einstein-Chern-Simons, que é uma teoria para gravidade em $(2+1)$ dimensões. Neste modelo, considerado um "toy model", os operadores que constituem a base de Barnes-Rivers não formam um conjunto fechado. Vimos que a estrutura antissimétrica do símbolo de Levi-Civita implicou na necessidade de obtenção de uma versão estendida para a base de Barnes-Rivers, que permitisse calcular corretamente o propagador num cenário de $(2+1)$ dimensões. O propagador obtido possui, além do polo usual, um polo adicional massivo que é compatível com a causalidade e pode tornar a teoria renormalizável [59].

No quinto e último capítulo, Consideramos uma teoria de gravitação constituída pelo acoplamento entre o campo bumblebee (b^{μ}) e a lagrangiana de Einstein-Hilbert. Aqui também se faz necessária uma extensão da base de Barnes-Rivers, através de projetores que contenham o campo bumblebee, que é o violador da simetria de Lorentz. Além do modo não-massivo, obtivemos mais dois modos de propagação, ambos possuem relações de dispersão que respeitam a causalidade. No entanto, a relação de dispersão associada ao modo $\boxdot = 0$ apresenta um comportamento físico inesperado, onde a energia passa a depender do ângulo entre a direção de propagação e o campo de fundo $(\mathbf{b})$, que é o responsável pela violação da simetria de Lorentz.

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APÊNDICE A

Variação do escalar de Ricci e de $\delta\left(\sqrt{-g}\right)$

Para calcular a variação do escalar de Ricci, vamos escrevê-lo na forma contraída $R = g^{\mu\nu}R_{\mu\nu}$, de modo que vale

$$\delta R = \delta\left(g^{\mu\nu}R_{\mu\nu}\right) = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\delta R_{\mu\nu}.$$

A variação do tensor de Ricci é dada pela *identidade de Palatini* [11],

$$\delta R_{\mu\nu} = \nabla_\lambda\left(\delta\Gamma^\lambda{}_{\mu\nu}\right) - \nabla_\nu\left(\delta\Gamma^\lambda{}_{\mu\lambda}\right), \tag{A.0.1}$$

sendo

$$\nabla_\lambda A^\mu = \partial_\lambda A^\mu + A^\kappa\Gamma^\mu{}_{\kappa\lambda}, \tag{A.0.2}$$

a derivada covariante. Podemos demonstrar a eq. [A.0.1] variando cada termo do tensor de Ricci

$$\delta R_{\mu\nu} = \partial_\lambda\delta\Gamma^\lambda{}_{\mu\nu} - \partial_\nu\delta\Gamma^\lambda{}_{\mu\lambda} + \delta\Gamma^\alpha{}_{\mu\nu}\Gamma^\lambda{}_{\alpha\lambda} - \delta\Gamma^\lambda{}_{\alpha\nu}\Gamma^\alpha{}_{\mu\lambda} + \Gamma^\alpha{}_{\mu\nu}\delta\Gamma^\lambda{}_{\alpha\lambda} - \Gamma^\lambda{}_{\alpha\nu}\delta\Gamma^\alpha{}_{\mu\lambda}. \tag{A.0.3}$$

Para calcular esta expressão, precisamos encontrar uma forma para o termo $\delta\Gamma^\lambda{}_{\mu\nu}$. Para isso, partimos da diferenciação da matriz inversa,

$$\delta g^{\lambda\gamma} = -g^{\rho\lambda}g^{\sigma\gamma}\delta g_{\rho\sigma}, \tag{A.0.4}$$

e os símbolos de Christoffel de primeiro tipo são dados por

$$\Gamma_{\mu\nu\gamma} = g_{\gamma\lambda}\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}\left(\partial_\nu g_{\mu\gamma} + \partial_\mu g_{\nu\gamma} - \partial_\gamma g_{\mu\nu}\right). \tag{A.0.5}$$

Podemos então escrever a variação dos símbolos de Christoffel como

$$\begin{aligned}
\delta\Gamma^\lambda{}_{\mu\nu} &= \delta\left(g^{\gamma\lambda}\Gamma_{\mu\nu\gamma}\right) = \Gamma_{\mu\nu\gamma}\delta g^{\gamma\lambda} + g^{\rho\lambda}\delta\Gamma_{\mu\nu\rho},\\
&= -g^{\sigma\gamma}g^{\rho\lambda}\delta g_{\rho\sigma}\Gamma_{\mu\nu\gamma} + g^{\rho\lambda}\delta\Gamma_{\mu\nu\rho},\\
&= -g^{\rho\lambda}\delta g_{\rho\sigma}\Gamma^\sigma{}_{\mu\nu} + \frac{1}{2}g^{\rho\lambda}\left(\partial_\nu\delta g_{\mu\rho} + \partial_\mu\delta g_{\nu\rho} - \partial_\rho\delta g_{\mu\nu}\right),\\
\delta\Gamma^\lambda{}_{\mu\nu} &= \frac{1}{2}g^{\rho\lambda}\left(\partial_\nu\delta g_{\mu\rho} + \partial_\mu\delta g_{\nu\rho} - \partial_\rho\delta g_{\mu\nu} - 2\Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma}\right).
\end{aligned} \tag{A.0.6}$$

Como $2\Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma} = \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma} + \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma}$, vamos então somar e subtrair este termo na equação acima, de modo que temos:

$$\begin{aligned}
\delta\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\big(&\partial_\nu\delta g_{\mu\rho} + \partial_\mu\delta g_{\nu\rho} - \partial_\rho\delta g_{\mu\nu} - \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma}\\
&-\Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma} + \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma} + \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma} - \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma} - \Gamma^\sigma{}_{\mu\nu}\delta g_{\rho\sigma}\big).
\end{aligned}$$

Renomeando os índices contraídos, surgem termos que podem ser identificados como a derivada covariante,

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(\partial_{\nu}\delta g_{\mu\rho} - \Gamma^{\alpha}{}_{\mu\nu}\delta g_{\rho\alpha} - \Gamma^{\alpha}{}_{\nu\rho}\delta g_{\alpha\mu}\right) + \left(\partial_{\mu}\delta g_{\nu\rho} - \Gamma^{\alpha}{}_{\nu\mu}\delta g_{\rho\alpha} - \Gamma^{\alpha}{}_{\rho\mu}\delta g_{\alpha\nu}\right)$$
$$- \left(\partial_{\rho}\delta g_{\mu\nu} - \Gamma^{\alpha}{}_{\mu\rho}\delta g_{\nu\alpha} - \Gamma^{\alpha}{}_{\nu\rho}\delta g_{\sigma\mu}\right),$$

$$\delta\Gamma^{\lambda}{}_{\mu\nu} = \frac{1}{2}g^{\rho\lambda}\left(\nabla_{\nu}\delta g_{\mu\rho} + \nabla_{\mu}\delta g_{\nu\rho} - \nabla_{\rho}\delta g_{\mu\nu}\right). \tag{A.0.7}$$

Importante frisar que estamos considerando a métrica como sendo uma matriz diagonal, por isso que as eqs. (A.0.4) e (A.0.13) não possuem problemas na sua concepção. Na eq. (A.0.7), desde que $\delta g_{\mu\rho}$ seja um tensor, $\delta\Gamma^{\lambda}{}_{\mu\nu}$ também será um tensor. Portanto, podemos usar a definição de derivada covariante para simplificar a expressão (A.0.3). Somando e subtraindo os termos $\Gamma^{\lambda}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\lambda}$ e $\Gamma^{\lambda}{}_{\mu\kappa}\Gamma^{\kappa}{}_{\nu\lambda}$, temos

$$\delta R_{\mu\nu} = \partial_{\lambda}\delta\Gamma^{\lambda}{}_{\mu\nu} + \delta\left(\Gamma^{\sigma}{}_{\mu\nu}\Gamma^{\lambda}{}_{\sigma\lambda}\right) - \delta\left(\Gamma^{\lambda}{}_{\alpha\nu}\Gamma^{\alpha}{}_{\mu\lambda}\right) - \delta\left(\Gamma^{\lambda}{}_{\mu\kappa}\Gamma^{\kappa}{}_{\nu\lambda}\right)$$
$$- \left[\partial_{\nu}\delta\Gamma^{\lambda}{}_{\mu\lambda} + \delta\left(\Gamma^{\sigma}{}_{\mu\nu}\Gamma^{\lambda}{}_{\sigma\lambda}\right) - \delta\left(\Gamma^{\alpha}{}_{\mu\nu}\Gamma^{\lambda}{}_{\alpha\lambda}\right) - \delta\left(\Gamma^{\lambda}{}_{\mu\kappa}\Gamma^{\kappa}{}_{\nu\lambda}\right)\right],$$

que pela definição de derivada covariante pode ser escrita como

$$\delta R_{\mu\nu} = \nabla_{\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\lambda}{}_{\mu\lambda}\right). \tag{A.0.8}$$

Logo,

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + g^{\mu\nu}\left[\nabla_{\lambda}\left(\delta\Gamma^{\lambda}{}_{\mu\nu}\right) - \nabla_{\nu}\left(\delta\Gamma^{\lambda}{}_{\mu\lambda}\right)\right].$$

A variação do termo $\delta\left(\sqrt{-g}\right)$ é

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2\sqrt{-g}}\delta g. \tag{A.0.9}$$

A variação de g pode ser escrita como

$$\delta g = \frac{\partial g}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma}, \tag{A.0.10}$$

e sabemos que o determinante de uma matriz pode ser obtido pela seguinte expressão

$$g = \det\left(g^{\rho\sigma}\right) = g^{\rho\sigma}A_{\rho\sigma}, \tag{A.0.11}$$

onde $A^{\rho\sigma}$ representa a matriz dos cofatores da matriz em questão. Assim, ao derivarmos a expressão acima temos:

$$\frac{\partial g}{\partial g_{\rho\sigma}} = A^{\rho\sigma}. \tag{A.0.12}$$

De modo que podemos reescrever a variação de g como:

$$\delta g = \frac{\partial g}{\partial g_{\rho\sigma}}\delta g_{\rho\sigma} = A^{\rho\sigma}\delta g_{\rho\sigma}. \tag{A.0.13}$$

Por outro lado, a inversa de uma matriz é dada pela expressão

$$\left(g_{\rho\sigma}\right)^{-1} = g^{\rho\sigma} = \frac{1}{g}\left(A^{\rho\sigma}\right)^{t},$$

ou ainda,

$$A^{\rho\sigma} = \left(gg^{\rho\sigma}\right)^t.$$

Logo,

$$\delta g = \left(gg^{\rho\sigma}\right)^t \delta g_{\rho\sigma},$$

como o tensor métrico é simétrico, a operação de transposição não causará alteração

$$\delta g = gg^{\rho\sigma}\delta g_{\rho\sigma}.$$

Finalmente,

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2\sqrt{-g}}gg^{\rho\sigma}\delta g_{\rho\sigma} = \frac{1}{2}\sqrt{-g}g^{\rho\sigma}\delta g_{\rho\sigma},$$

por meio da eq. (A.0.4), temos

$$\delta\left(\sqrt{-g}\right) = -\frac{1}{2}\sqrt{-g}g_{\rho\sigma}\delta g^{\rho\sigma}. \tag{A.0.14}$$

Assim,

$$\begin{aligned}\delta\left(\sqrt{-g}R\right) &= \sqrt{-g}\delta R + R\delta\left(\sqrt{-g}\right)\\ &= \sqrt{-g}\left[R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\lambda\left(g^{\mu\nu}\delta\Gamma^\lambda{}_{\mu\nu}\right) - \nabla_\nu\left(g^{\mu\nu}\delta\Gamma^\lambda{}_{\mu\lambda}\right) - \frac{1}{2}Rg_{\mu\nu}\delta g^{\mu\nu}\right],\end{aligned}$$

onde usamos o fato de que $\nabla_\lambda g^{\mu\nu} = 0$ e $g_{\mu\nu}\delta g^{\mu\nu} = -g^{\mu\nu}\delta g_{\mu\nu}$.

APÊNDICE B

Expansão em segunda ordem da lagrangiana de Einstein-Hilbert

Vimos que a versão linearizada da lagrangiana de Einstein-Hilbert possui apenas termos em primeira ordem no campo do gráviton. Para encontrarmos a sua expansão adequada, vamos buscar pelos termos de até segunda ordem para cada um dos entes que compõem a lagrangiana

$$\mathcal{S}_{E-H} = \int \sqrt{-g\left[g_{\mu\nu}\right]} R\left[g_{\mu\nu}\right] dx^4. \tag{B.0.1}$$

A expansão a ser adotada para a métrica é a seguinte

$$g_{\mu\nu} = \eta_{\mu\nu} + \varepsilon h_{\mu\nu}. \tag{B.0.2}$$

O primeiro passo é obter a métrica inversa e a raiz do determinante da métrica. Para pequenos valores de ε teremos

$$|\varepsilon h_{\mu\nu}| \ll 1, \tag{B.0.3}$$

e podemos aplicar a série de Newmann

$$(I - A)^{-1} = \sum_{l=0}^{\infty} A^l.$$

A fim de obter uma expressão para comparação, podemos reescrever

$$g_{\mu\nu} = \eta_{\mu\kappa}\left(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu}\right), \tag{B.0.4}$$

assim

$$A = -\varepsilon h^{\kappa}{}_{\nu}. \tag{B.0.5}$$

Aplicando a série de Newmann

$$(I - A)^{-1} = I + A + A^2 + ..., \tag{B.0.6}$$

logo,

$$\begin{aligned} g^{\mu\nu} &= \eta^{\mu\kappa}\left(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu}\right)^{-1} \\ g^{\mu\nu} &= \eta^{\mu\kappa}\left[\delta^{\nu}{}_{\kappa} - \varepsilon h^{\nu}{}_{\kappa} + \varepsilon^2 h^{\nu}{}_{\gamma} h^{\gamma}{}_{\kappa} + \mathcal{O}\left(\varepsilon^3\right)\right], \\ &= \eta^{\mu\nu} - \varepsilon h^{\mu\nu} + \varepsilon^2 h^{\mu\gamma} h^{\nu}{}_{\gamma} + \mathcal{O}\left(\varepsilon^3\right). \end{aligned}$$

A seguir, usamos a expressão acima para obter a expansão do determinante da métrica, a identidade que nos permite fazer isso é a seguinte

$$\det(A) = \exp\left[Tr\log(A)\right]. \tag{B.0.7}$$

Assim,

$$\sqrt{-\det(g_{\mu\nu})} = \sqrt{-\det(\eta_{\mu\kappa})}\sqrt{\det(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu})}, \tag{B.0.8}$$

sabemos que $\det(\eta_{\mu\kappa}) = -1$, e $\det(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu}) = \exp\left[Tr\log(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu})\right]$

$$\begin{aligned}\sqrt{-\det(g_{\mu\nu})} &= \sqrt{\exp\left[Tr\log(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu})\right]}, \\ &= \exp\left[\frac{1}{2}Tr\log(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu})\right], \end{aligned} \tag{B.0.9}$$

onde usamos $\sqrt{e^a} = e^{\frac{a}{2}}$. A expansão em séries da função logarítmica de uma matriz é dada por

$$\log(A) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(A-I)^n}{n}, \tag{B.0.10}$$

o termo I é matriz identidade que nesse caso é representada pela delta de Kronecker, assim

$$\begin{aligned}\log(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu}) &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\delta^{\kappa}{}_{\nu} + \varepsilon h^{\kappa}{}_{\nu} - \delta^{\kappa}{}_{\nu})^n}{n}, \\ &= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\varepsilon h^{\kappa}{}_{\nu})^n}{n}, \end{aligned} \tag{B.0.11}$$

logo

$$\sqrt{-\det(g_{\mu\nu})} = \exp\left[\frac{1}{2}Tr\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\varepsilon h^{\kappa}{}_{\nu})^n}{n}\right], \tag{B.0.12}$$

usando a expansão em séries da exponencial

$$\begin{aligned}\sqrt{-\det(g_{\mu\nu})} &= \sum_{k=0}^{\infty}\frac{1}{k!}\left[\frac{1}{2}Tr\sum_{n=1}^{\infty}\frac{(-1)^{n-1}(\varepsilon h^{\kappa}{}_{\nu})^n}{n}\right]^k, \\ &= 1 + \frac{1}{2}\varepsilon h + \frac{1}{8}\varepsilon^2\left(h^2 - 2h^{\kappa}{}_{\gamma}h^{\gamma}{}_{\nu}\right) + \mathcal{O}\left(\varepsilon^3\right). \end{aligned} \tag{B.0.13}$$

Essa passagem para a última linha é a única que ainda não está completamente clara, no entanto, obter o restante dos elementos é bem intuitivo. Para os Símbolos de Chirstoffel

$$\begin{aligned}\Gamma^{\alpha}{}_{\mu\nu} &= \frac{1}{2}g^{\alpha\gamma}\left(\partial_{\mu}g_{\gamma\nu} + \partial_{\nu}g_{\gamma\mu} - \partial_{\gamma}g_{\mu\nu}\right), \\ &= \frac{1}{2}\left[\eta^{\alpha\gamma} - \varepsilon h^{\alpha\gamma} + \varepsilon^2 h^{\alpha\kappa}h^{\gamma}{}_{\kappa} + \mathcal{O}\left(\varepsilon^3\right)\right]\left(\varepsilon\partial_{\mu}h_{\gamma\nu} + \varepsilon\partial_{\nu}h_{\gamma\mu} - \varepsilon\partial_{\gamma}h_{\mu\nu}\right), \end{aligned} \tag{B.0.14}$$

onde usamos o fato de a métrica de Minkowski ser constante, consequentemente $\partial_{\gamma}\eta_{\mu\nu} = 0$, o que resulta

$$\begin{aligned}\Gamma^{\alpha}{}_{\mu\nu} &= \frac{\varepsilon}{2}\left(\partial_{\mu}h^{\alpha}{}_{\nu} + \partial_{\nu}h^{\alpha}{}_{\mu} - \partial^{\alpha}h_{\mu\nu}\right) - \frac{\varepsilon^2}{2}h^{\alpha}{}_{\gamma}\left(\partial_{\mu}h^{\gamma}{}_{\nu} + \partial_{\nu}h^{\gamma}{}_{\mu} - \partial^{\gamma}h_{\mu\nu}\right) + \mathcal{O}\left(\varepsilon^3\right), \\ \Gamma^{\alpha}{}_{\mu\nu} &= \varepsilon\mathcal{G}^{\alpha}{}_{\mu\nu} - \varepsilon^2 h^{\alpha}{}_{\gamma}\mathcal{G}^{\gamma}{}_{\mu\nu} + \mathcal{O}\left(\varepsilon^3\right), \end{aligned} \tag{B.0.15}$$

definimos a quantidade

$$\mathcal{G}^{\alpha}{}_{\mu\nu} = \frac{1}{2}\left(\partial_\mu h^{\alpha}{}_{\nu} + \partial_\nu h^{\alpha}{}_{\mu} - \partial^{\alpha} h_{\mu\nu}\right), \tag{B.0.16}$$

que será útil na simplificação das expressões a seguir. O tensor de Riemann, em termos dos Simbolos de Christoffel, possui a forma

$$R^{\alpha}{}_{\mu\gamma\nu} = \partial_\gamma \Gamma^{\alpha}{}_{\mu\nu} - \partial_\nu \Gamma^{\alpha}{}_{\mu\gamma} + \Gamma^{\alpha}{}_{\gamma\kappa}\Gamma^{\kappa}{}_{\mu\nu} - \Gamma^{\alpha}{}_{\nu\kappa}\Gamma^{\kappa}{}_{\mu\gamma}, \tag{B.0.17}$$

assim o tensor de Ricci $R_{\mu\nu}$ pode ser escrito como

$$R_{\mu\nu} = -2\partial_{[\nu}\Gamma^{\gamma}{}_{\gamma]\mu} + 2\Gamma^{\kappa}{}_{\mu[\nu}\Gamma^{\alpha}{}_{\alpha]\kappa}, \tag{B.0.18}$$

os colchetes nos índices fazem parte da notação compacta para antissimetria. Substituindo os Símbolos de Christoffel obtidos anteriormente, podemos obter

$$R_{\mu\nu} = -2\varepsilon\partial_{[\mu}\mathcal{G}^{\gamma}{}_{\gamma]\nu} + 2\varepsilon^2\mathcal{G}^{\kappa}{}_{\nu[\mu}\mathcal{G}^{\alpha}{}_{\alpha]\kappa}. \tag{B.0.19}$$

Finalmente, podemos obter uma expressão para o escalar de Ricci que é o termo que compõe lagrangiana de Einstein-Hilbert

$$\begin{aligned} R &= g^{\mu\nu}R_{\mu\nu}, \\ &= \left[\eta^{\mu\nu} - \varepsilon h^{\mu\nu} + \varepsilon^2 h^{\mu\gamma}h^{\nu}{}_{\gamma} + \mathcal{O}\left(\varepsilon^3\right)\right] R_{\mu\nu}, \end{aligned} \tag{B.0.20}$$

os termos que irão contribuir são

$$\begin{aligned} \eta^{\mu\nu}R_{\mu\nu} &= -2\varepsilon\eta^{\mu\nu}\partial_{[\mu}\mathcal{G}^{\gamma}{}_{\gamma]\nu} + 2\varepsilon^2\eta^{\mu\nu}\mathcal{G}^{\kappa}{}_{\nu[\mu}\mathcal{G}^{\alpha}{}_{\alpha]\kappa}, \\ \varepsilon h^{\mu\nu}R_{\mu\nu} &= 2\varepsilon^2 h^{\mu\nu}\partial_{[\mu}\mathcal{G}^{\gamma}{}_{\gamma]\nu}. \end{aligned} \tag{B.0.21}$$

Escrevemos os resultados termo a termo

$$\eta^{\mu\nu}\partial_{[\mu}\mathcal{G}^{\gamma}{}_{\gamma]\nu} = \frac{1}{2}\Box h - \frac{1}{2}\partial_\mu\partial_\nu h^{\mu\nu}, \tag{B.0.22}$$

$$\eta^{\mu\nu}\mathcal{G}^{\kappa}{}_{\nu[\mu}\mathcal{G}^{\alpha}{}_{\alpha]\kappa} = \frac{1}{4}\partial_\mu h\partial_\nu h^{\mu\nu} - \frac{1}{8}\partial_\alpha h\partial^\alpha h - \frac{1}{4}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha} + \frac{1}{8}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha}, \tag{B.0.23}$$

$$h^{\mu\nu}\partial_{[\mu}\mathcal{G}^{\gamma}{}_{\gamma]\nu} = \frac{1}{4}h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}h^{\mu\nu}\partial_\gamma\partial_\mu h^{\gamma}{}_{\nu} + \frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu}. \tag{B.0.24}$$

Assim,

$$\begin{aligned} R = -2\varepsilon&\left(\frac{1}{2}\Box h - \frac{1}{2}\partial_\mu\partial_\nu h^{\mu\nu}\right) \\ &+ 2\varepsilon^2\left(\frac{1}{4}\partial_\mu h\partial_\nu h^{\mu\nu} - \frac{1}{8}\partial_\alpha h\partial^\alpha h - \frac{1}{4}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha} + \frac{1}{8}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha}\right) \\ &\qquad + 2\varepsilon^2\left(\frac{1}{4}h^{\mu\nu}\partial_\mu\partial_\nu h - \frac{1}{2}h^{\mu\nu}\partial_\gamma\partial_\mu h^{\gamma}{}_{\nu} + \frac{1}{4}h^{\mu\nu}\Box h_{\mu\nu}\right) + \mathcal{O}\left(\varepsilon^3\right), \end{aligned}$$

usando o truque da derivada total, simplificamos a expressão acima

$$R = -\varepsilon\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right) + 2\varepsilon^2\left(-\frac{1}{8}\partial_\alpha h\partial^\alpha h - \frac{1}{8}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha} + \frac{1}{4}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha}\right) + \mathcal{O}\left(\varepsilon^3\right). \tag{B.0.25}$$

A lagrangiana de Einstein-Hilbert, tomada sob a forma

$$\mathcal{L}_{E-H} = \sqrt{-g\left[g_{\mu\nu}\right]}R\left[g_{\mu\nu}\right], \tag{B.0.26}$$

na sua expansão em segunda ordem pode ser obtida ao substituir as expansões para $\sqrt{-g}$ e R

$$\mathcal{L}_{E-H} = \left[1 + \frac{1}{2}\varepsilon h + \frac{1}{8}\varepsilon^2\left(h^2 - 2h^{\kappa}{}_{\gamma}h^{\gamma}{}_{\nu}\right) + \mathcal{O}\left(\varepsilon^3\right)\right]$$
$$\times\left[-\varepsilon\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right) + 2\varepsilon^2\left(-\frac{1}{8}\partial_\alpha h\partial^\alpha h - \frac{1}{8}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha} + \frac{1}{4}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha}\right) + \mathcal{O}\left(\varepsilon^3\right)\right], \tag{B.0.27}$$

$$\mathcal{L}_{E-H} = -\varepsilon\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right) - \frac{1}{2}\varepsilon^2\left(h\Box h - h\partial_\mu\partial_\nu h^{\mu\nu}\right)$$
$$+\varepsilon^2\left(-\frac{1}{4}\partial_\alpha h\partial^\alpha h - \frac{1}{4}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha} + \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha}\right) + \mathcal{O}\left(\varepsilon^3\right), \tag{B.0.28}$$

$$\mathcal{L}_{E-H} = -\varepsilon\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right)$$
$$+\varepsilon^2\left(-\frac{1}{4}\partial_\alpha h\partial^\alpha h - \frac{1}{4}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha} + \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha} - \frac{1}{2}h\Box h + \frac{1}{2}h\partial_\mu\partial_\nu h^{\mu\nu}\right) + \mathcal{O}\left(\varepsilon^3\right), \tag{B.0.29}$$

novamente, podemos simplificar reescrevendo alguns termos usando a derivada total

$$\mathcal{L}_{E-H} = -\varepsilon\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right)$$
$$+\varepsilon^2\left(\frac{1}{4}\partial_\alpha h\partial^\alpha h - \frac{1}{2}\partial_\mu h\partial_\nu h^{\mu\nu} - \frac{1}{4}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha} + \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha}\right) + \mathcal{O}\left(\varepsilon^3\right). \tag{B.0.30}$$

Podemos identificar esses resultados da seguinte forma:

$$\mathcal{L}^{(1)}_{E-H} = -\left(\Box h - \partial_\mu\partial_\nu h^{\mu\nu}\right), \tag{B.0.31}$$

é o termo da expansão em primeira ordem, também conhecida como aproximação para campo gravitacional fraco, utilizada principalmente na obtenção das ondas gravitacionais. O segundo,

$$\mathcal{L}^{(2)}_{E-H} = \frac{1}{4}\partial_\alpha h\partial^\alpha h - \frac{1}{2}\partial_\mu h\partial_\nu h^{\mu\nu} - \frac{1}{4}\partial^\nu h_{\mu\alpha}\partial_\nu h^{\mu\alpha} + \frac{1}{2}\partial^\alpha h^{\mu\nu}\partial_\nu h_{\mu\alpha}, \tag{B.0.32}$$

será usado na obtenção do propagador. Uma observação importante é que fizemos a expansão na métrica de Minkowski, ou seja, o termo $\mathcal{L}^{(0)}_{E-H}$ não aparece porque é nulo devido ao fato de a métrica de Minkowski ser constante.

APÊNDICE C

Equações de campo em primeira ordem em $h_{\mu\nu}$

Seguindo a abordagem do Schutz [19], partimos da expansão

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \text{ com } |h_{\mu\nu}| \ll 1, \tag{C.0.1}$$

onde também temos

$$g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu}. \tag{C.0.2}$$

De forma direta, a linearização das equações de campo consiste em escrever as equações de Einstein em termos da métrica (C.0.1), ou seja, devemos escrever as contrações do tensor de Riemann (tensor e escalar de Ricci) em termos da eq. (C.0.1). Vimos que o tensor de Riemann pode ser escrito em termos dos símbolos de Christoffel:

$$R^{\nu}{}_{\alpha\beta\mu} = \partial_{\beta}\Gamma^{\nu}{}_{\alpha\mu} - \partial_{\mu}\Gamma^{\nu}{}_{\alpha\beta} + \Gamma^{\nu}{}_{\beta\gamma}\Gamma^{\gamma}{}_{\alpha\mu} - \Gamma^{\nu}{}_{\mu\gamma}\Gamma^{\gamma}{}_{\alpha\beta}, \tag{C.0.3}$$

enquanto os símbolos de Christoffel são:

$$\Gamma^{\nu}{}_{\alpha\mu} = \frac{1}{2}g^{\gamma\nu}\left(\partial_{\mu}g_{\gamma\alpha} + \partial_{\alpha}g_{\gamma\mu} - \partial_{\gamma}g_{\alpha\mu}\right). \tag{C.0.4}$$

Substituindo as equações (C.0.1) e (C.0.2) em (C.0.4), temos:

$$\Gamma^{\nu}{}_{\alpha\mu} = \frac{1}{2}\left(\eta^{\gamma\nu} - h^{\gamma\nu}\right)\left[\partial_{\mu}\left(\eta_{\gamma\alpha} + h_{\gamma\alpha}\right) + \partial_{\alpha}\left(\eta_{\gamma\mu} + h_{\gamma\mu}\right) - \partial_{\gamma}\left(\eta_{\alpha\mu} + h_{\alpha\mu}\right)\right].$$

Usando o fato de a métrica de Minkowski ser constante, temos:

$$\Gamma^{\nu}{}_{\alpha\mu} = \frac{1}{2}\left(\eta^{\gamma\nu} - h^{\gamma\nu}\right)\left(\partial_{\mu}h_{\gamma\alpha} + \partial_{\alpha}h_{\gamma\mu} - \partial_{\gamma}h_{\alpha\mu}\right).$$

Vamos considerar apenas termos em primeira ordem em $|h_{\mu\nu}|$, ou seja:

$$\Gamma^{\nu}{}_{\alpha\mu} = \frac{1}{2}\eta^{\gamma\nu}\left(\partial_{\mu}h_{\gamma\alpha} + \partial_{\alpha}h_{\gamma\mu} - \partial_{\gamma}h_{\alpha\mu}\right). \tag{C.0.5}$$

Observe que, ao substituírmos (C.0.5) em (C.0.3), os produtos de dois símbolos de Christoffel geram apenas termos de segunda ordem em $|h_{\mu\nu}|$, isso reduz nosso trabalho apenas ao cálculo dos dois primeiros termos do tensor de Riemann, a saber:

$$R^{\nu}{}_{\alpha\beta\mu} = \frac{1}{2}\eta^{\gamma\nu}\left[\partial_{\beta}\left(\partial_{\mu}h_{\gamma\alpha} + \partial_{\alpha}h_{\gamma\mu} - \partial_{\gamma}h_{\alpha\mu}\right)\right] - \frac{1}{2}\eta^{\gamma\nu}\left[\partial_{\mu}\left(\partial_{\beta}h_{\gamma\alpha} + \partial_{\alpha}h_{\gamma\beta} - \partial_{\gamma}h_{\alpha\beta}\right)\right],$$

$$R^{\nu}{}_{\alpha\beta\mu} = \frac{1}{2}\eta^{\gamma\nu}\left(\partial_\beta\partial_\alpha h_{\gamma\mu} - \partial_\beta\partial_\gamma h_{\alpha\mu} - \partial_\mu\partial_\alpha h_{\gamma\beta} + \partial_\mu\partial_\gamma h_{\alpha\beta}\right). \tag{C.0.6}$$

Lembrando que a métrica de Minkowski assumiu o papel de abaixar/levantar índices tensoriais. De fato:

$$R_{\nu\alpha\beta\mu} = \eta_{\nu\lambda}R^{\lambda}{}_{\alpha\beta\mu} = \frac{1}{2}\underbrace{\eta_{\nu\lambda}\eta^{\gamma\lambda}}_{\delta^\gamma_\nu}\left(\partial_\beta\partial_\alpha h_{\gamma\mu} - \partial_\beta\partial_\gamma h_{\alpha\mu} - \partial_\mu\partial_\alpha h_{\gamma\beta} + \partial_\mu\partial_\gamma h_{\alpha\beta}\right),$$

resultando em:

$$R_{\nu\alpha\beta\mu} = \frac{1}{2}\left(\partial_\beta\partial_\alpha h_{\nu\mu} - \partial_\beta\partial_\nu h_{\alpha\mu} - \partial_\mu\partial_\alpha h_{\nu\beta} + \partial_\mu\partial_\nu h_{\alpha\beta}\right).$$

Essa é a expressão para o tensor de Riemann num regime de campo gravitacional fraco. Entretanto, ainda nos falta expressar o tensor e o escalar de Ricci, dado a seguir:

$$R_{\alpha\mu} = \eta^{\nu\beta}R_{\nu\alpha\beta\mu}, \tag{C.0.7}$$

logo,

$$R_{\alpha\mu} = \frac{1}{2}\eta^{\nu\beta}\left(\partial_\beta\partial_\alpha h_{\nu\mu} - \partial_\beta\partial_\nu h_{\alpha\mu} - \partial_\mu\partial_\alpha h_{\nu\beta} + \partial_\mu\partial_\nu h_{\alpha\beta}\right), \tag{C.0.8}$$

ou ainda,

$$R_{\alpha\mu} = \frac{1}{2}\left(\partial_\beta\partial_\alpha h^{\beta}{}_{\mu} - \Box h_{\alpha\mu} - \partial_\mu\partial_\alpha h + \partial_\mu\partial_\nu h_{\alpha}{}^{\nu}\right), \tag{C.0.9}$$

onde $\Box = \partial^\beta\partial_\beta$, e

$$h = \eta^{\nu\beta}h_{\nu\beta} \text{ ou } h = h^{\beta}{}_{\beta}.$$

A partir de $(C.0.9)$, podemos obter o escalar de Ricci $R = \eta^{\alpha\mu}R_{\alpha\mu}$,

$$R = \frac{1}{2}\eta^{\alpha\mu}\left(\partial_\beta\partial_\alpha h^{\beta}{}_{\mu} - \Box h_{\alpha\mu} - \partial_\mu\partial_\alpha h + \partial_\mu\partial_\nu h_{\alpha}{}^{\nu}\right),$$

$$R = \frac{1}{2}\left(\partial_\beta\partial_\alpha h^{\beta\alpha} - \Box h - \Box h + \partial_\mu\partial_\nu h^{\mu\nu}\right).$$

Fazendo uma mudança nos índices contraídos do primeiro termo, escrevemos:

$$R = \partial_\mu\partial_\nu h^{\mu\nu} - \Box h. \tag{C.0.10}$$

Esse é o escalar de Ricci para o caso de um campo gravitacional fraco. Por meio das equações $(C.0.9)$ e $(C.0.10)$, podemos construir o tensor de Einstein covariante. Substituindo $(C.0.9)$ e $(C.0.10)$ nas equações de movimento de Einstein,

$$\begin{aligned}
G_{\mu\nu} &= \frac{1}{2}\left(\partial_\beta\partial_\nu h^{\beta}{}_{\mu} - \Box h_{\nu\mu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\gamma h_{\nu}{}^{\gamma}\right) - \frac{1}{2}\left(\eta_{\mu\nu} + h_{\mu\nu}\right)\left(\partial_\rho\partial_\lambda h^{\rho\lambda} - \Box h\right),\\
&= \frac{1}{2}\left(\partial_\beta\partial_\nu h^{\beta}{}_{\mu} - \Box h_{\nu\mu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\gamma h_{\nu}{}^{\gamma}\right) - \frac{1}{2}\eta_{\mu\nu}\left(\partial_\rho\partial_\lambda h^{\rho\lambda} - \Box h\right),
\end{aligned}$$

$$G_{\mu\nu} = \frac{1}{2}\left(\partial_\beta\partial_\nu h^{\beta}{}_{\mu} - \Box h_{\nu\mu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\gamma h_{\nu}{}^{\gamma} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h\right). \tag{C.0.11}$$

Assim, as equações de Einstein em primeira ordem são:

$$\partial_\beta\partial_\nu h^{\beta}{}_{\mu} - \Box h_{\nu\mu} - \partial_\mu\partial_\nu h + \partial_\mu\partial_\gamma h_{\nu}{}^{\gamma} - \eta_{\mu\nu}\partial_\rho\partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h = 2\kappa T_{\mu\nu}. \tag{C.0.12}$$

APÊNDICE D

Cálculo da contração $S_{\mu\nu,\kappa\lambda}S^{\kappa\lambda}{}_{,\alpha\beta}$

Temos

$$S_{\mu\nu,\kappa\lambda}S^{\kappa\lambda}{}_{,\alpha\beta} = \frac{\partial^\sigma}{2}\left(\epsilon_{\mu\sigma\kappa}\theta_{\nu\lambda} + \epsilon_{\mu\sigma\lambda}\theta_{\nu\kappa} + \epsilon_{\nu\sigma\kappa}\theta_{\mu\lambda} + \epsilon_{\nu\sigma\lambda}\theta_{\mu\kappa}\right) \times$$
$$\frac{\partial^\varpi}{2}\left(\epsilon^\kappa{}_{\varpi\alpha}\theta^\lambda{}_\beta + \epsilon^\kappa{}_{\varpi\beta}\theta^\lambda{}_\alpha + \epsilon^\lambda{}_{\varpi\alpha}\theta^\kappa{}_\beta + \epsilon^\lambda{}_{\varpi\beta}\theta^\kappa{}_\alpha\right),$$

após desenvolver todos os produtos, vamos usar o seguinte resultado nas parcelas obtidas:

$$\begin{aligned}
\partial^\sigma\epsilon_{\mu\sigma\kappa}\partial^\varpi\epsilon^\kappa{}_{\varpi\alpha} &= \partial^\sigma\partial_\varpi\eta_{\alpha\varpi}\epsilon_{\mu\sigma\kappa}\epsilon^{\kappa\varpi\alpha} \\
&= \partial_\alpha\partial_\lambda - \partial^\sigma\partial_\sigma\eta_{\alpha\lambda} \\
&= -\Box\theta_{\alpha\lambda}.
\end{aligned}$$

Após agrupar os termos semelhantes

$$\begin{aligned}
S_{\mu\nu,\kappa\lambda}S^{\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{4}4\left(-\Box\theta_{\mu\beta}\theta_{\nu\alpha} - \Box\theta_{\nu\beta}\theta_{\mu\alpha}\right) + \frac{\partial^\sigma\partial^\varpi}{4}4\left(\epsilon_{\mu\sigma\beta}\epsilon_{\nu\varpi\alpha} + \epsilon_{\nu\sigma\beta}\epsilon_{\mu\varpi\alpha}\right), \\
&= -2\Box\left(P^{(0-\theta)} + P^{(2)}\right) + \partial^\sigma\partial^\varpi\left(\epsilon_{\mu\sigma\beta}\epsilon_{\nu\varpi\alpha} + \epsilon_{\nu\sigma\beta}\epsilon_{\mu\varpi\alpha}\right),
\end{aligned}$$

reescrevemos os produtos dos símbolos de Levi-Civita na segunda parcela, sob a forma

$$\epsilon_{\mu\sigma\beta}\epsilon_{\nu\varpi\alpha} = \eta_{\mu\nu}\left(\eta_{\sigma\varpi}\eta_{\alpha\beta} - \eta_{\sigma\alpha}\eta_{\beta\varpi}\right) - \eta_{\mu\varpi}\left(\eta_{\sigma\nu}\eta_{\alpha\beta} - \eta_{\sigma\alpha}\eta_{\beta\nu}\right) + \eta_{\mu\alpha}\left(\eta_{\sigma\nu}\eta_{\varpi\beta} - \eta_{\sigma\varpi}\eta_{\beta\nu}\right),$$

o que permite expressar:

$$\begin{aligned}
\partial^\sigma\partial^\varpi\left(\epsilon_{\mu\sigma\beta}\epsilon_{\nu\varpi\alpha} + \epsilon_{\nu\sigma\beta}\epsilon_{\mu\varpi\alpha}\right) = \Box&\left[2\eta_{\mu\nu}\eta_{\alpha\beta} - 2\left(\eta_{\mu\nu}\omega_{\alpha\beta} + \omega_{\mu\nu}\eta_{\alpha\beta}\right)\right. \\
&\left.+\left(\eta_{\beta\nu}\omega_{\alpha\mu} + \omega_{\nu\alpha}\eta_{\beta\mu} + \eta_{\nu\alpha}\omega_{\mu\beta} + \omega_{\nu\beta}\eta_{\mu\alpha}\right) - \left(\eta_{\mu\alpha}\eta_{\nu\beta} + \eta_{\alpha\nu}\eta_{\mu\beta}\right)\right].
\end{aligned}$$

Por meio das identidades dos operadores da base de Barnes-Rivers, podemos escrever

$$\begin{aligned}
\partial^\sigma\partial^\varpi\left(\epsilon_{\mu\sigma\beta}\epsilon_{\nu\varpi\alpha} + \epsilon_{\nu\sigma\beta}\epsilon_{\mu\varpi\alpha}\right) &= \Box\left[2\left(2P^{(0-\theta)} + P^{(0-\omega)} + P^{(0-\theta\omega)}\right) - 2\left(2P^{(0-\omega)} + P^{(0-\theta\omega)}\right)\right. \\
&\left.+2\left(P^{(1)} + P^{(0-\omega)}\right) + 2\left(P^{(1)} + P^{(0-\omega)}\right) - 2\left(P^{(0-\theta)} + P^{(0-\omega)} + P^{(1)} + P^{(2)}\right)\right], \\
&= \Box\left(2P^{(0-\theta)} - 2P^{(2)}\right).
\end{aligned}$$

Substituindo, finalmente resulta:

$$\begin{aligned} S_{\mu\nu,\kappa\lambda}S^{\kappa\lambda}{}_{,\alpha\beta} &= -2\Box\left(P^{(0-\theta)} + P^{(2)}\right) + \Box\left(2P^{(0-\theta)} - 2P^{(2)}\right) \\ &= -4\Box P^{(2)}. \end{aligned}$$

Note que usamos $\eta_{\mu\nu}\eta_{\alpha\beta} = 2P^{(0-\theta)} + P^{(0-\omega)} + P^{(0-\theta\omega)}$, ao invés de $\eta_{\mu\nu}\eta_{\alpha\beta} = 3P^{(0-\theta)} + P^{(0-\omega)} + P^{(0-\theta\omega)}$. Os números 3 e 2 destes resultados são apenas fatores relacionados à dimensão do espaço-tempo trabalhado.

APÊNDICE E

Cálculo das contrações da base estendida

Para $\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right) \times \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta},$$

$$= \frac{1}{3}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\theta^{\kappa\lambda}\theta_{\alpha\beta}, \tag{E.0.1}$$

$$= \frac{1}{3}\left(\theta^{\lambda}{}_{\mu}\Lambda_{\nu\lambda} + \theta^{\kappa}{}_{\nu}\Lambda_{\mu\kappa}\right), \tag{E.0.2}$$

a contração $\theta^{\lambda}{}_{\mu}\Lambda_{\nu\lambda}$ pode ser simplesmente calculada

$$\theta^{\lambda}{}_{\mu}\Lambda_{\nu\lambda} = \left(\delta^{\lambda}{}_{\mu} - \omega^{\lambda}{}_{\mu}\right)b_\nu b_\lambda,$$

$$= b_\nu b_\mu - \frac{(b\cdot p)}{p^2}p_\mu b_\nu,$$

nos permitindo escrever

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\left[b_\nu b_\mu - \frac{(b\cdot p)}{p^2}p_\mu b_\nu + b_\nu b_\mu - \frac{(b\cdot p)}{p^2}p_\nu b_\mu\right]\theta_{\alpha\beta},$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{3}\left[2\Lambda_{\mu\nu} - \frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\right]\theta_{\alpha\beta},$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}. \tag{E.0.3}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{3}\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

O próximo é $\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\tilde{\Sigma}_{\mu\nu}\theta_{\kappa\lambda} + \tilde{\Sigma}_{\kappa\lambda}\theta_{\mu\nu}\right) \times \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}, \tag{E.0.4}$$

a contração $\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa\lambda}$ é nula, pois $\theta^{\kappa\lambda}p_\kappa = 0$,

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}. \tag{E.0.5}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

Contraindo com $\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}$

$$\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta},\\
&= \frac{1}{3\sqrt{3}}\left[\theta_{\mu\nu}\left(b^2-\frac{(b\cdot p)^2}{p^2}\right)+3\Lambda_{\mu\nu}\right]\theta_{\alpha\beta},\\
&= \frac{1}{3\sqrt{3}}\left(b^2-\frac{(b\cdot p)^2}{p^2}\right)\theta_{\mu\nu}\theta_{\alpha\beta}+\frac{1}{\sqrt{3}}\theta_{\alpha\beta}\Lambda_{\mu\nu},\\
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} &= \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta}+\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta},
\end{aligned} \tag{E.0.6}$$

onde usamos o resultado $\theta^{\kappa\lambda}\Lambda_{\kappa\lambda} = b^2-(b\cdot p)^2/p^2$.

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}.$$

Prosseguiremos com o restante

$$\begin{aligned}
\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} &= \Lambda_{\mu\nu}\Lambda_{\kappa\lambda}\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta},\\
&= \frac{b^2p^2-(b\cdot p)^2}{3p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned} \tag{E.0.7}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \left(\omega_{\mu\kappa}\Lambda_{\nu\lambda}+\omega_{\mu\lambda}\Lambda_{\nu\kappa}+\omega_{\nu\kappa}\Lambda_{\mu\lambda}+\omega_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta} = 0, \tag{E.0.8}$$

pois $\theta^{\kappa\lambda}\omega_{\mu\kappa} = 0$.

$$\begin{aligned}
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\Lambda_{\kappa\lambda}+\omega_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta},\\
&= \frac{b^2p^2-(b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned} \tag{E.0.9}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta} = \frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} = \left(\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta} = 0. \tag{E.0.10}$$

$$\begin{aligned}
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta} &= \left(\Lambda_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta},\\
&= \frac{b^2p^2-(b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned} \tag{E.0.11}$$

$$P^{(0-\theta)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

Agora calcularemos as contrações de $P^{(1)}_{\mu\nu,\alpha\beta}$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}+\theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}+\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right), \tag{E.0.12}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{4}\Big[&\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right)+\theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa}\left(\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right)\\&+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right)+\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\left(\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right)\Big]\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{4}\Big[&\theta_{\mu\alpha}\left(\tilde{\Sigma}_{\nu\lambda}\omega^{\lambda}{}_{\beta}+\tilde{\Sigma}_{\nu\kappa}\omega^{\kappa}{}_{\beta}\right)+\theta_{\mu\beta}\left(\tilde{\Sigma}_{\nu\lambda}\omega^{\lambda}{}_{\alpha}+\tilde{\Sigma}_{\nu\kappa}\omega^{\kappa}{}_{\alpha}\right)\\&+\theta_{\nu\alpha}\left(\tilde{\Sigma}_{\mu\lambda}\omega^{\lambda}{}_{\beta}+\tilde{\Sigma}_{\mu\kappa}\omega^{\kappa}{}_{\beta}\right)+\theta_{\nu\beta}\left(\tilde{\Sigma}_{\mu\lambda}\omega^{\lambda}{}_{\alpha}+\tilde{\Sigma}_{\mu\kappa}\omega^{\kappa}{}_{\alpha}\right)\Big]\end{aligned}$$

observe que

$$\begin{aligned}\tilde{\Sigma}_{\nu\lambda}\omega^{\lambda}{}_{\beta}+\tilde{\Sigma}_{\nu\kappa}\omega^{\kappa}{}_{\beta}&=\left(b_\nu p_\lambda+b_\lambda p_\nu\right)\omega^{\lambda}{}_{\beta}+\left(b_\nu p_\kappa+b_\kappa p_\nu\right)\omega^{\kappa}{}_{\beta},\\&=2b_\nu p_\beta+2\left(b\cdot p\right)\omega_{\nu\beta},\end{aligned}$$

assim,

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\frac{1}{2}\{&\theta_{\mu\alpha}\left[b_\nu p_\beta+(b\cdot p)\,\omega_{\nu\beta}\right]+\theta_{\mu\beta}\left[b_\nu p_\alpha+(b\cdot p)\,\omega_{\nu\alpha}\right]\\&+\theta_{\nu\alpha}\left[b_\mu p_\beta+(b\cdot p)\,\omega_{\mu\beta}\right]+\theta_{\nu\beta}\left[b_\mu p_\alpha+(b\cdot p)\,\omega_{\mu\alpha}\right]\}\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=&\frac{1}{2}\left(b\cdot p\right)\left(\theta_{\mu\alpha}\omega_{\nu\beta}+\theta_{\mu\beta}\omega_{\nu\alpha}+\theta_{\nu\alpha}\omega_{\mu\beta}+\theta_{\nu\beta}\omega_{\mu\alpha}\right)\\&+\frac{1}{2}\left(\theta_{\mu\alpha}b_\nu p_\beta+\theta_{\mu\beta}b_\nu p_\alpha+\theta_{\nu\alpha}b_\mu p_\beta+\theta_{\nu\beta}b_\mu p_\alpha\right),\end{aligned}$$

a primeira parcela é $P^{(1)}_{\mu\nu,\alpha\beta}$ e para a segunda iremos usar a notação da Ref. [38] $b_\nu p_\beta=\Sigma_{\nu\beta}$, logo

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta}=\left(b\cdot p\right)P^{(1)}_{\mu\nu,\alpha\beta}+\Pi^{(1-a)}_{\mu\nu,\alpha\beta}, \tag{E.0.13}$$

onde

$$\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta}=\Pi^{(1-a)}_{\mu\nu,\alpha\beta}+\Pi^{(1-b)}_{\mu\nu,\alpha\beta},$$

sendo

$$\Pi^{(1-a)}_{\mu\nu,\alpha\beta}=\frac{1}{2}\left(\theta_{\mu\alpha}b_\nu p_\beta+\theta_{\mu\beta}b_\nu p_\alpha+\theta_{\nu\alpha}b_\mu p_\beta+\theta_{\nu\beta}b_\mu p_\alpha\right), \tag{E.0.14}$$

$$\Pi^{(1-b)}_{\mu\nu,\alpha\beta}=\frac{1}{2}\left(\theta_{\mu\alpha}b_\beta p_\nu+\theta_{\mu\beta}b_\alpha p_\nu+\theta_{\nu\alpha}b_\beta p_\mu+\theta_{\nu\beta}b_\alpha p_\mu\right). \tag{E.0.15}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta}=\left(b\cdot p\right)P^{(1)}_{\mu\nu,\alpha\beta}+\Pi^{(1-b)}_{\mu\nu,\alpha\beta}.$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),$$
$$= \frac{1}{2}\left[\theta_{\mu\kappa}\Lambda_{\nu\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right)+\theta_{\nu\kappa}\Lambda_{\mu\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right)\right],$$
$$= \frac{1}{2}\left[\left(\theta_{\mu\alpha}b_{\nu}p_{\beta}+\theta_{\mu\beta}b_{\nu}p_{\alpha}\right)\frac{(b\cdot p)}{p^2}+\left(\theta_{\nu\alpha}b_{\mu}p_{\beta}+\theta_{\nu\beta}b_{\mu}p_{\alpha}\right)\frac{(b\cdot p)}{p^2}\right],$$
$$= \frac{1}{2}\left(\theta_{\mu\alpha}b_{\nu}p_{\beta}+\theta_{\mu\beta}b_{\nu}p_{\alpha}+\theta_{\nu\alpha}b_{\mu}p_{\beta}+\theta_{\nu\beta}b_{\mu}p_{\alpha}\right)\frac{(b\cdot p)}{p^2},$$
$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)}{p^2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta}. \tag{E.0.16}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)}{p^2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta}.$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),$$
$$= \frac{1}{2\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),$$
$$= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\left(\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right), \tag{E.0.17}$$

$$\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} = \left(b_{\nu}p_{\lambda}+b_{\lambda}p_{\nu}\right)\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}$$
$$= b_{\alpha}p_{\lambda}\omega^{\lambda}{}_{\beta}-(b\cdot p)\,\omega_{\alpha\lambda}\omega^{\lambda}{}_{\beta},$$
$$= b_{\alpha}p_{\beta}-(b\cdot p)\,\omega_{\alpha\beta},$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\left[b_{\alpha}p_{\beta}+b_{\beta}p_{\alpha}-2\,(b\cdot p)\,\omega_{\alpha\beta}\right],$$
$$= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{2}{\sqrt{3}}\,(b\cdot p)\,\theta_{\mu\nu}\omega_{\alpha\beta}. \tag{E.0.18}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}-\frac{2}{\sqrt{3}}\,(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),$$
$$= \frac{1}{2\sqrt{3}}\theta_{\mu\nu}\Lambda_{\kappa\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),$$
$$= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\left(b_{\alpha}p_{\beta}+b_{\beta}p_{\alpha}-(b\cdot p)\,\omega_{\alpha\beta}-(b\cdot p)\,\omega_{\alpha\beta}\right)\frac{(b\cdot p)}{p^2},$$
$$= \frac{(b\cdot p)}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta}. \tag{E.0.19}$$

$$P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}-2\frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta}.$$

$$
\begin{aligned}
\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= \Lambda_{\mu\nu}\Lambda_{\kappa\lambda} \times \frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right), \\
&= \Lambda_{\mu\nu}\left(\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right), \\
&= \Lambda_{\mu\nu}\left(b_\alpha p_\beta - (b\cdot p)\,\omega_{\alpha\beta} + b_\beta p_\alpha - (b\cdot p)\,\omega_{\alpha\beta}\right)\frac{(b\cdot p)}{p^2}, \\
&= \frac{(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\frac{(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}. \qquad \text{(E.0.20)}
\end{aligned}
$$

$$
P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - 2\frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}.
$$

$$
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \left(\omega_{\mu\kappa}\Lambda_{\nu\lambda} + \omega_{\mu\lambda}\Lambda_{\nu\kappa} + \omega_{\nu\kappa}\Lambda_{\mu\lambda} + \omega_{\nu\lambda}\Lambda_{\mu\kappa}\right) \times \frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),
$$

$$
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \Lambda_{\nu\lambda}\theta^{\lambda}{}_{\alpha}\omega_{\mu\beta} + \Lambda_{\nu\lambda}\theta^{\lambda}{}_{\beta}\omega_{\mu\alpha} + \Lambda_{\mu\lambda}\theta^{\lambda}{}_{\alpha}\omega_{\nu\beta} + \Lambda_{\mu\lambda}\theta^{\lambda}{}_{\beta}\omega_{\nu\alpha},
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= b_\nu b_\alpha\omega_{\mu\beta} + b_\nu b_\beta\omega_{\mu\alpha} + b_\mu b_\alpha\omega_{\nu\beta} + b_\mu b_\beta\omega_{\nu\alpha} \\
&\quad - \left(b_\nu p_\alpha\omega_{\mu\beta} + b_\nu p_\beta\omega_{\mu\alpha} + b_\mu p_\alpha\omega_{\nu\beta} + b_\mu p_\beta\omega_{\nu\alpha}\right)\frac{(b\cdot p)}{p^2}, \qquad \text{(E.0.21)}
\end{aligned}
$$

$$
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}. \qquad \text{(E.0.22)}
$$

$$
P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}{}_{,\alpha\beta} = \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}. \qquad \text{(E.0.23)}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\Lambda_{\kappa\lambda} + \omega_{\kappa\lambda}\Lambda_{\mu\nu}\right) \times \frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right), \\
&= \frac{1}{2}\left(\omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta} + \omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha} + \omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right), \\
&= \omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta} + \omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}, \\
&= \omega_{\mu\nu}\left(\frac{(b\cdot p)}{p^2}b_\alpha p_\beta - \frac{(b\cdot p)^2}{p^2}\omega_{\alpha\beta} + \frac{(b\cdot p)}{p^2}b_\beta p_\alpha - \frac{(b\cdot p)^2}{p^2}\omega_{\alpha\beta}\right), \\
&= \omega_{\mu\nu}\left(b_\alpha p_\beta + b_\beta p_\alpha\right)\frac{(b\cdot p)}{p^2} - \frac{2\,(b\cdot p)^2}{p^2}\omega_{\alpha\beta}\omega_{\mu\nu}, \\
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= \frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2\,(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}. \qquad \text{(E.0.24)}
\end{aligned}
$$

$$
P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - \frac{2\,(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right),\\
&= \frac{1}{2}\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right)\\
&= \omega_{\mu\nu}\left(\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}\right),\\
&= \omega_{\mu\nu}\left[b_{\alpha}p_{\beta}-(b\cdot p)\,\omega_{\alpha\beta}+b_{\beta}p_{\alpha}-(b\cdot p)\,\omega_{\alpha\beta}\right],\\
&= \tilde{\Sigma}_{\alpha\beta}\omega_{\mu\nu}-2\,(b\cdot p)\,\omega_{\alpha\beta}\omega_{\mu\nu},\\
&= \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta}.
\end{aligned}
\tag{E.0.25}
$$

$$
P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}-2\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta}.
$$

$$
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} = \left(\Lambda_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\omega^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\omega^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\omega^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\omega^{\kappa}{}_{\alpha}\right), \tag{E.0.26}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= \Lambda_{\mu\nu}\left(b_{\alpha}p_{\beta}-(b\cdot p)\,\omega_{\alpha\beta}+b_{\beta}p_{\alpha}-(b\cdot p)\,\omega_{\alpha\beta}\right)\\
&\quad+\tilde{\Sigma}_{\mu\nu}\left(\frac{(b\cdot p)}{p^2}b_{\alpha}p_{\beta}-\frac{(b\cdot p)^2}{p^2}\omega_{\alpha\beta}+\frac{(b\cdot p)}{p^2}b_{\beta}p_{\alpha}-\frac{(b\cdot p)^2}{p^2}\omega_{\alpha\beta}\right),
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(1)\kappa\lambda}{}_{,\alpha\beta} &= \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\,(b\cdot p)\,\omega_{\alpha\beta}\Lambda_{\mu\nu}+\frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\alpha\beta}\tilde{\Sigma}_{\mu\nu}-\frac{2\,(b\cdot p)^2}{p^2}\omega_{\alpha\beta}\tilde{\Sigma}_{\mu\nu},\\
&= \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-2\,(b\cdot p)\,\Lambda_{\mu\nu}\omega_{\alpha\beta}+(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\frac{2\,(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned}
\tag{E.0.27}
$$

$$
P^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta}+(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-\frac{2\,(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
$$

O terceiro projetor é $P^{(2)}_{\mu\nu,\alpha\beta}$

$$
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}+\theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}+\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\times\left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right] \tag{E.0.28}
$$

note que

$$
\begin{aligned}
\tilde{\Sigma}_{\nu\lambda}\theta^{\lambda}{}_{\beta} &= \left(b_{\nu}p_{\lambda}+b_{\lambda}p_{\nu}\right)\theta^{\lambda}{}_{\beta}\\
&= b_{\lambda}p_{\nu}\left(\eta^{\lambda}{}_{\beta}-\omega^{\lambda}{}_{\beta}\right),\\
&= b_{\beta}p_{\nu}-(b\cdot p)\,\omega_{\nu\beta}
\end{aligned}
$$

Portanto,

$$
\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{2}\left[\theta_{\mu\alpha}\left(b_{\beta}p_{\nu}-(b\cdot p)\,\omega_{\nu\beta}\right)+2\theta_{\mu\beta}\left(b_{\alpha}p_{\nu}-(b\cdot p)\,\omega_{\nu\alpha}\right)\right.\\
&\quad\left.+2\theta_{\nu\alpha}\left(b_{\beta}p_{\mu}-(b\cdot p)\,\omega_{\mu\beta}\right)+2\theta_{\nu\beta}\left(b_{\alpha}p_{\mu}-(b\cdot p)\,\omega_{\mu\alpha}\right)\right]\\
&\quad-\frac{1}{6}\left[2\left(b_{\mu}p_{\nu}-(b\cdot p)\,\omega_{\mu\nu}\right)+2\left(b_{\nu}p_{\mu}-(b\cdot p)\,\omega_{\mu\nu}\right)\right]\theta_{\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2} &\left(\theta_{\mu\alpha} b_\beta p_\nu + \theta_{\mu\beta} b_\alpha p_\nu + \theta_{\nu\alpha} b_\beta p_\mu + \theta_{\nu\beta} b_\alpha p_\mu\right) \\
&- \frac{1}{2} \left(b\cdot p\right) \left(\omega_{\nu\beta}\theta_{\mu\alpha} + \omega_{\nu\alpha}\theta_{\mu\beta} + \omega_{\mu\beta}\theta_{\nu\alpha} + \omega_{\mu\alpha}\theta_{\nu\beta}\right) \\
&- \frac{1}{3} \left[\theta_{\alpha\beta} b_\mu p_\nu + \theta_{\alpha\beta} b_\nu p_\mu - \left(b\cdot p\right)\left(\omega_{\mu\nu}\theta_{\alpha\beta} + \omega_{\mu\nu}\theta_{\alpha\beta}\right)\right],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2} &\left(\theta_{\mu\alpha} b_\beta p_\nu + \theta_{\mu\beta} b_\alpha p_\nu + \theta_{\nu\alpha} b_\beta p_\mu + \theta_{\nu\beta} b_\alpha p_\mu\right) \\
&- \frac{1}{2} \left(b\cdot p\right) \left(\omega_{\nu\beta}\theta_{\mu\alpha} + \omega_{\nu\alpha}\theta_{\mu\beta} + \omega_{\mu\beta}\theta_{\nu\alpha} + \omega_{\mu\alpha}\theta_{\nu\beta}\right) \\
&- \frac{1}{3} \left[\left(b_\mu p_\nu + b_\nu p_\mu\right)\theta_{\alpha\beta} - 2\left(b\cdot p\right)\omega_{\mu\nu}\theta_{\alpha\beta}\right],
\end{aligned}
$$

$$
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \Pi^{(1-b)}_{\mu\nu,\alpha\beta} - \left(b\cdot p\right) P^{(1)}_{\mu\nu,\alpha\beta} - \frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{3}\left(b\cdot p\right)\omega_{\mu\nu}\theta_{\alpha\beta}. \tag{E.0.29}
$$

$$
P^{(2)}_{\mu\nu,\kappa\lambda} \tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta} = \Pi^{(1-a)}_{\mu\nu,\alpha\beta} - \left(b\cdot p\right) P^{(1)}_{\mu\nu,\alpha\beta} - \frac{1}{3}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{3}\left(b\cdot p\right)\theta_{\mu\nu}\omega_{\alpha\beta}.
$$

$$
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right) \times \left[\frac{1}{2}\left(\theta^\kappa{}_\alpha\theta^\lambda{}_\beta + \theta^\lambda{}_\alpha\theta^\kappa{}_\beta\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right], \tag{E.0.30}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2} &\left[\theta_{\mu\alpha}\left(b_\nu b_\beta - \frac{(b\cdot p)}{p^2} b_\nu p_\beta\right) + \theta_{\mu\beta}\left(b_\nu b_\alpha - \frac{(b\cdot p)}{p^2} b_\nu p_\alpha\right)\right. \\
&\left. + \theta_{\nu\alpha}\left(b_\mu b_\beta - \frac{(b\cdot p)}{p^2} b_\mu p_\beta\right) + \theta_{\nu\beta}\left(b_\mu b_\alpha - \frac{(b\cdot p)}{p^2} b_\mu p_\alpha\right)\right] \\
&- \frac{1}{3}\left[\left(b_\nu b_\mu - \frac{(b\cdot p)}{p^2} b_\nu p_\mu\right)\theta_{\alpha\beta} + \left(b_\nu b_\mu - \frac{(b\cdot p)}{p^2} b_\mu p_\nu\right)\theta_{\alpha\beta}\right],
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2} &\left(\theta_{\mu\alpha} b_\nu b_\beta + \theta_{\nu\alpha} b_\mu b_\beta + \theta_{\mu\beta} b_\nu b_\alpha + \theta_{\nu\beta} b_\mu b_\alpha\right) \\
&- \frac{(b\cdot p)}{2p^2}\left(b_\nu p_\beta\theta_{\mu\alpha} + b_\nu p_\alpha\theta_{\mu\beta} + b_\mu p_\beta\theta_{\nu\alpha} + b_\mu p_\alpha\theta_{\nu\beta}\right) \\
&- \frac{1}{3}\left(b_\nu b_\mu + b_\nu b_\mu\right)\theta_{\alpha\beta} + \frac{(b\cdot p)}{3p^2}\left(b_\nu p_\mu + b_\mu p_\nu\right)\theta_{\alpha\beta},
\end{aligned}
$$

$$
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta} - \frac{2}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{(b\cdot p)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}. \tag{E.0.31}
$$

$$
P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta} = \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta} - \frac{2}{3}\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{(b\cdot p)}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
$$

$$
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right) \times \left[\frac{1}{2}\left(\theta^\kappa{}_\alpha\theta^\lambda{}_\beta + \theta^\lambda{}_\alpha\theta^\kappa{}_\beta\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right],
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} &\left[\frac{1}{2}\left(\theta^\kappa{}_\alpha\theta^\lambda{}_\beta + \theta^\lambda{}_\alpha\theta^\kappa{}_\beta\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right] \\
&+ \frac{1}{\sqrt{3}}\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\left[\frac{1}{2}\left(\theta^\kappa{}_\alpha\theta^\lambda{}_\beta + \theta^\lambda{}_\alpha\theta^\kappa{}_\beta\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right],
\end{aligned}
$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2\sqrt{3}}\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3\sqrt{3}}\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\theta^{\kappa\lambda}\theta_{\alpha\beta},$$
$$= \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} - \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} = 0. \quad \text{(E.0.32)}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right], \quad \text{(E.0.33)}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\kappa\lambda}\left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right]$$
$$+\frac{1}{\sqrt{3}}\theta_{\kappa\lambda}\Lambda_{\mu\nu}\left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right], \quad \text{(E.0.34)}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} - \frac{1}{3\sqrt{3}}\theta_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\kappa\lambda}\Lambda_{\mu\nu}\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} - \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}, \quad \text{(E.0.35)}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b_{\alpha}b_{\beta}\theta_{\mu\nu} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\alpha\beta}\theta_{\mu\nu} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\alpha\beta}\theta_{\mu\nu}$$
$$-\frac{1}{3\sqrt{3}}\theta_{\mu\nu}\theta_{\alpha\beta}\left[b^2-\frac{(b\cdot p)^2}{p^2}\right] + \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}, \quad \text{(E.0.36)}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta}.$$
$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta} - \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta}.$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \Lambda_{\mu\nu}\Lambda_{\kappa\lambda}\times\left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right],$$
$$= \frac{1}{2}\Lambda_{\kappa\lambda}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)\Lambda_{\mu\nu} - \frac{1}{3}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}\Lambda_{\mu\nu}\theta_{\alpha\beta},$$
$$= \Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}\Lambda_{\mu\nu} - \frac{b^2p^2-(b\cdot p)^2}{3p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta}$$
$$= \Lambda_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2-(b\cdot p)^2}{3p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta},$$
$$= \tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2-(b\cdot p)^2}{3p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta}. \quad \text{(E.0.37)}$$

$$P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} = \tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}P^{(2)\kappa\lambda}{}_{,\alpha\beta} = \left(\omega_{\mu\kappa}\Lambda_{\nu\lambda}+\omega_{\mu\lambda}\Lambda_{\nu\kappa}+\omega_{\nu\kappa}\Lambda_{\mu\lambda}+\omega_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right)-\frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right],$$
$$= 0.$$

$$
\begin{aligned}
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\Lambda_{\kappa\lambda} + \omega_{\kappa\lambda}\Lambda_{\mu\nu}\right) \times \left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right], \\
&= \frac{1}{2}\omega_{\mu\nu}\Lambda_{\kappa\lambda}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3}\omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}\theta_{\alpha\beta}, \\
&= \omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta}, \\
&= \omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta}, \\
&= \omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned}
$$

$$
P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta} = \Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right) \times \left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right], \\
&= 0.
\end{aligned}
\tag{E.0.38}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda} P^{(2)\kappa\lambda}{}_{,\alpha\beta} &= \left(\Lambda_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right) \times \left[\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3}\theta^{\kappa\lambda}\theta_{\alpha\beta}\right], \\
&= \frac{1}{2}\Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\left(\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\theta^{\kappa}{}_{\beta}\right) - \frac{1}{3}\Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\theta^{\kappa\lambda}\theta_{\alpha\beta}, \\
&= \tilde{\Sigma}_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa}{}_{\alpha}\theta^{\lambda}{}_{\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \\
&= \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}, \\
&= \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - (b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned}
\tag{E.0.39}
$$

$$
P^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - (b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
$$

Para $P^{(0-\omega)}_{\mu\nu,\alpha\beta}$:

$$
\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \omega^{\kappa\lambda}\omega_{\alpha\beta}, \\
&= 0.
\end{aligned}
\tag{E.0.40}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right) \times \omega^{\kappa\lambda}\omega_{\alpha\beta}, \\
&= 0.
\end{aligned}
\tag{E.0.41}
$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}} \left(\theta_{\mu\nu} \tilde{\Sigma}_{\kappa\lambda} + \theta_{\kappa\lambda} \tilde{\Sigma}_{\mu\nu} \right) \times \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \frac{1}{\sqrt{3}} \theta_{\mu\nu} \tilde{\Sigma}_{\kappa\lambda} \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \frac{2}{\sqrt{3}} (b \cdot p) \, \theta_{\mu\nu} \omega_{\alpha\beta}. \tag{E.0.42}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} \tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{\sqrt{3}} (b \cdot p) \, \omega_{\mu\nu} \theta_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}} \left(\theta_{\mu\nu} \Lambda_{\kappa\lambda} + \theta_{\kappa\lambda} \Lambda_{\mu\nu} \right) \times \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \frac{1}{\sqrt{3}} \theta_{\mu\nu} \Lambda_{\kappa\lambda} \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \frac{(b \cdot p)^2}{\sqrt{3} p^2} \theta_{\mu\nu} \omega_{\alpha\beta}. \tag{E.0.43}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} \tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b \cdot p)^2}{\sqrt{3} p^2} \omega_{\mu\nu} \theta_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \Lambda_{\mu\nu} \Lambda_{\kappa\lambda} \times \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \frac{(b \cdot p)^2}{p^2} \Lambda_{\mu\nu} \omega_{\alpha\beta}. \tag{E.0.44}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} \tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b \cdot p)^2}{p^2} \omega_{\mu\nu} \Lambda_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \left(\omega_{\mu\kappa} \Lambda_{\nu\lambda} + \omega_{\mu\lambda} \Lambda_{\nu\kappa} + \omega_{\nu\kappa} \Lambda_{\mu\lambda} + \omega_{\nu\lambda} \Lambda_{\mu\kappa} \right) \times \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \left(\omega^{\lambda}{}_{\mu} \Lambda_{\nu\lambda} + \omega^{\kappa}{}_{\mu} \Lambda_{\nu\kappa} + \omega^{\lambda}{}_{\nu} \Lambda_{\mu\lambda} + \omega^{\kappa}{}_{\nu} \Lambda_{\mu\kappa} \right) \omega_{\alpha\beta},$$
$$= 2 \left(\omega^{\lambda}{}_{\mu} \Lambda_{\nu\lambda} + \omega^{\kappa}{}_{\nu} \Lambda_{\mu\kappa} \right) \omega_{\alpha\beta},$$
$$= \frac{2 (b \cdot p)}{p^2} \left(b_\mu p_\nu + b_\nu p_\mu \right) \omega_{\alpha\beta},$$
$$= \frac{2 (b \cdot p)}{p^2} \tilde{\Sigma}_{\mu\nu} \omega_{\alpha\beta}. \tag{E.0.45}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda} \tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}{}_{,\alpha\beta} = \frac{2 (b \cdot p)}{p^2} \omega_{\mu\nu} \tilde{\Sigma}_{\alpha\beta}.$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda} P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} = \left(\omega_{\mu\nu} \Lambda_{\kappa\lambda} + \omega_{\kappa\lambda} \Lambda_{\mu\nu} \right) \times \omega^{\kappa\lambda} \omega_{\alpha\beta},$$
$$= \omega_{\mu\nu} \Lambda_{\kappa\lambda} \omega^{\kappa\lambda} \omega_{\alpha\beta} + \omega_{\kappa\lambda} \omega^{\kappa\lambda} \Lambda_{\mu\nu} \omega_{\alpha\beta},$$
$$= \frac{(b \cdot p)^2}{p^2} \omega_{\mu\nu} \omega_{\alpha\beta} + \Lambda_{\mu\nu} \omega_{\alpha\beta},$$
$$= \frac{(b \cdot p)^2}{p^2} P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \Lambda_{\mu\nu} \omega_{\alpha\beta}. \tag{E.0.46}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \omega_{\mu\nu}\Lambda_{\alpha\beta}$$

$$\begin{aligned}\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\omega^{\kappa\lambda}\omega_{\alpha\beta},\\ &= \tilde{\Sigma}_{\kappa\lambda}\omega^{\kappa\lambda}\omega_{\mu\nu}\omega_{\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\\ &= 2\,(b\cdot p)\,\omega_{\mu\nu}\omega_{\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\\ &= 2\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.\end{aligned} \tag{E.0.47}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\omega)\kappa\lambda}{}_{,\alpha\beta} &= \left(\Lambda_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\omega^{\kappa\lambda}\omega_{\alpha\beta},\\ &= 2\,(b\cdot p)\,\Lambda_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.\end{aligned} \tag{E.0.48}$$

$$P^{(0-\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

Prosseguimos com $P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta}$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda} + \theta^{\kappa\lambda}\omega_{\alpha\beta}\right), \tag{E.0.49}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{2\sqrt{3}}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\theta^{\kappa\lambda}\omega_{\alpha\beta},$$

note que esse resultado deve ser análogo à $\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta)\kappa\lambda}{}_{,\alpha\beta}$,

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\omega_{\alpha\beta},\\ &= \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - \frac{2}{\sqrt{3}}\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta}.\end{aligned} \tag{E.0.50}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}{}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2}{\sqrt{3}}\,(b\cdot p)\,P^{(0-\omega)}_{\mu\nu,\alpha\beta}.$$

$$\begin{aligned}\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda} + \theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\ &= \frac{1}{2\sqrt{3}}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\theta^{\kappa\lambda}\omega_{\alpha\beta},\\ &= \frac{1}{\sqrt{3}}\left(\theta^{\lambda}{}_{\mu}\Lambda_{\nu\lambda} + \theta^{\kappa}{}_{\nu}\Lambda_{\mu\kappa}\right)\omega_{\alpha\beta},\\ &= \frac{1}{\sqrt{3}}\left[\Lambda_{\mu\nu} - \frac{(b\cdot p)}{p^2}b_\nu p_\mu + \Lambda_{\mu\nu} - \frac{(b\cdot p)}{p^2}b_\mu p_\nu\right]\omega_{\alpha\beta},\\ &= \frac{1}{\sqrt{3}}\left[2\Lambda_{\mu\nu} - \frac{(b\cdot p)}{p^2}\left(b_\nu p_\mu + b_\mu p_\nu\right)\right]\omega_{\alpha\beta},\\ &= \frac{2}{\sqrt{3}}\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.\end{aligned} \tag{E.0.51}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}{}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda} + \theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right) \times \frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda} + \theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\
&= \frac{1}{3}\left(\theta_{\mu\nu}\theta_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\omega^{\kappa\lambda} + \theta_{\kappa\lambda}\theta^{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right),\\
&= \frac{2}{3}(b\cdot p)\,\theta_{\mu\nu}\theta_{\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\\
&= 2\,(b\cdot p)\,P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned} \tag{E.0.52}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}{}_{,\alpha\beta} = 2\,(b\cdot p)\,P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda} + \theta_{\kappa\lambda}\Lambda_{\mu\nu}\right) \times \frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda} + \theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\
&= \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}\Lambda_{\kappa\lambda}\omega^{\kappa\lambda} + \frac{1}{3}\theta_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}\omega_{\alpha\beta} + \Lambda_{\mu\nu}\omega_{\alpha\beta},\\
&= \frac{(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{3}\left(b^2 - \frac{(b\cdot p)^2}{p^2}\right)\theta_{\mu\nu}\omega_{\alpha\beta} + \Lambda_{\mu\nu}\omega_{\alpha\beta},\\
&= \frac{(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta} + \Lambda_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned} \tag{E.0.53}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\omega_{\mu\nu}\theta_{\alpha\beta} + \omega_{\mu\nu}\Lambda_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \Lambda_{\mu\nu}\Lambda_{\kappa\lambda} \times \frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda} + \theta^{\kappa\lambda}\omega_{\alpha\beta}\right).\\
&= \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}\Lambda_{\kappa\lambda}\omega^{\kappa\lambda} + \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\omega_{\alpha\beta}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda},\\
&= \frac{(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned} \tag{E.0.54}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}{}_{,\alpha\beta} = \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\kappa}\Lambda_{\nu\lambda} + \omega_{\mu\lambda}\Lambda_{\nu\kappa} + \omega_{\nu\kappa}\Lambda_{\mu\lambda} + \omega_{\nu\lambda}\Lambda_{\mu\kappa}\right) \times \frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda} + \theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\left(\omega_{\mu\kappa}\Lambda_{\nu\lambda} + \omega_{\mu\lambda}\Lambda_{\nu\kappa} + \omega_{\nu\kappa}\Lambda_{\mu\lambda} + \omega_{\nu\lambda}\Lambda_{\mu\kappa}\right)\theta_{\alpha\beta}\omega^{\kappa\lambda},\\
&= \frac{2}{\sqrt{3}}\left(\omega^{\lambda}{}_{\mu}\Lambda_{\nu\lambda} + \omega^{\kappa}{}_{\nu}\Lambda_{\mu\kappa}\right)\theta_{\alpha\beta},\\
&= \frac{2\,(b\cdot p)}{\sqrt{3}p^2}\left(b_\nu p_\mu + b_\mu p_\nu\right)\theta_{\alpha\beta},\\
&= \frac{2\,(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned} \tag{E.0.55}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}{}_{,\alpha\beta}=\frac{2\,(b\cdot p)}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= (\omega_{\mu\nu}\Lambda_{\kappa\lambda}+\omega_{\kappa\lambda}\Lambda_{\mu\nu})\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\
&=\frac{1}{\sqrt{3}}\omega_{\mu\nu}\theta_{\alpha\beta}\Lambda_{\kappa\lambda}\omega^{\kappa\lambda}+\frac{1}{\sqrt{3}}\omega_{\mu\nu}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}\omega_{\alpha\beta}+\frac{1}{\sqrt{3}}\omega_{\kappa\lambda}\omega^{\kappa\lambda}\Lambda_{\mu\nu}\theta_{\alpha\beta},\\
&=\frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\omega_{\alpha\beta}+\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta},\\
&=\frac{(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}+\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned}\tag{E.0.56}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}{}_{,\alpha\beta}=\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}P^{(0-\omega)}_{\mu\nu,\alpha\beta}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \left(\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\
&=\frac{1}{\sqrt{3}}\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}\theta_{\alpha\beta}\omega^{\kappa\lambda}+\frac{1}{\sqrt{3}}\omega_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}\theta^{\kappa\lambda}\omega_{\alpha\beta}+\frac{1}{\sqrt{3}}\omega_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}\omega^{\kappa\lambda},\\
&=\frac{2}{\sqrt{3}}\,(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.
\end{aligned}\tag{E.0.57}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\frac{2}{\sqrt{3}}\,(b\cdot p)\,\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}P^{(0-\theta\omega)\kappa\lambda}{}_{,\alpha\beta} &= \left(\Lambda_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\Lambda_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\omega^{\kappa\lambda}+\theta^{\kappa\lambda}\omega_{\alpha\beta}\right),\\
&=\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\theta_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\omega^{\kappa\lambda}+\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}\Lambda_{\kappa\lambda}\omega^{\kappa\lambda}+\frac{1}{\sqrt{3}}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta},\\
&=\frac{2}{\sqrt{3}}\,(b\cdot p)\,\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned}\tag{E.0.58}$$

$$P^{(0-\theta\omega)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}{}_{,\alpha\beta}=\frac{2}{\sqrt{3}}\,(b\cdot p)\,\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

Agora, apresentaremos as contrações que envolvem $\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=\frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}+\theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}+\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\tilde{\Sigma}^{\lambda}{}_{\beta}+\theta^{\lambda}{}_{\alpha}\tilde{\Sigma}^{\kappa}{}_{\beta}+\theta^{\kappa}{}_{\beta}\tilde{\Sigma}^{\lambda}{}_{\alpha}+\theta^{\lambda}{}_{\beta}\tilde{\Sigma}^{\kappa}{}_{\alpha}\right),\tag{E.0.59}$$

assim,

$$\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta}=\frac{1}{2}\Big(&\theta_{\mu\kappa}\theta^{\kappa}{}_{\alpha}\tilde{\Sigma}_{\nu\lambda}\tilde{\Sigma}^{\lambda}{}_{\beta}+\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\theta^{\lambda}{}_{\alpha}\tilde{\Sigma}^{\kappa}{}_{\beta}+\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\theta^{\kappa}{}_{\beta}\tilde{\Sigma}^{\lambda}{}_{\alpha}+\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\theta^{\lambda}{}_{\beta}\tilde{\Sigma}^{\kappa}{}_{\alpha}\\
&+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\theta^{\kappa}{}_{\alpha}\tilde{\Sigma}^{\lambda}{}_{\beta}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\theta^{\lambda}{}_{\alpha}\tilde{\Sigma}^{\kappa}{}_{\beta}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\theta^{\kappa}{}_{\beta}\tilde{\Sigma}^{\lambda}{}_{\alpha}+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\theta^{\lambda}{}_{\beta}\tilde{\Sigma}^{\kappa}{}_{\alpha}\Big),
\end{aligned}$$

observe que

$$\tilde{\Sigma}_{\nu\lambda}\tilde{\Sigma}^{\lambda}{}_{\beta} = \left(b_\nu p_\lambda + b_\lambda p_\nu\right)\left(b_\beta p^\lambda + b^\lambda p_\beta\right), \\ = p^2 b_\nu b_\beta + (b\cdot p)\left(b_\nu p_\beta + b_\beta p_\nu\right) + b^2 p_\nu p_\beta, \\ = p^2\Lambda_{\nu\beta} + (b\cdot p)\,\tilde{\Sigma}_{\nu\beta} + b^2p^2\omega_{\nu\beta},$$

então

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big[\theta_{\mu\alpha}\left(p^2\Lambda_{\nu\beta} + (b\cdot p)\,\tilde{\Sigma}_{\nu\beta} + b^2p^2\omega_{\nu\beta}\right) + \left(b_\mu p_\beta - (b\cdot p)\,\omega_{\mu\beta}\right)\left(b_\alpha p_\nu - (b\cdot p)\,\omega_{\nu\alpha}\right) \\ + \theta_{\mu\beta}\left(p^2\Lambda_{\nu\alpha} + (b\cdot p)\,\tilde{\Sigma}_{\nu\alpha} + b^2p^2\omega_{\nu\alpha}\right) + \left(b_\mu p_\alpha - (b\cdot p)\,\omega_{\alpha\mu}\right)\left(b_\beta p_\nu - (b\cdot p)\,\omega_{\nu\beta}\right) \\ + \theta_{\nu\alpha}\left(p^2\Lambda_{\mu\beta} + (b\cdot p)\,\tilde{\Sigma}_{\mu\beta} + b^2p^2\omega_{\mu\beta}\right) + \left(b_\nu p_\beta - (b\cdot p)\,\omega_{\nu\beta}\right)\left(b_\alpha p_\mu - (b\cdot p)\,\omega_{\mu\alpha}\right) \\ +\theta_{\nu\beta}\left(p^2\Lambda_{\mu\alpha} + (b\cdot p)\,\tilde{\Sigma}_{\mu\alpha} + b^2p^2\omega_{\mu\alpha}\right) + \left(b_\nu p_\alpha - (b\cdot p)\,\omega_{\alpha\nu}\right)\left(b_\beta p_\mu - (b\cdot p)\,\omega_{\mu\beta}\right)\Big],$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big[\theta_{\mu\alpha}\left(p^2\Lambda_{\nu\beta} + (b\cdot p)\,\tilde{\Sigma}_{\nu\beta} + b^2p^2\omega_{\nu\beta}\right) + \theta_{\mu\beta}\left(p^2\Lambda_{\nu\alpha} + (b\cdot p)\,\tilde{\Sigma}_{\nu\alpha} + b^2p^2\omega_{\nu\alpha}\right) \\ + \theta_{\nu\beta}\left(p^2\Lambda_{\mu\alpha} + (b\cdot p)\,\tilde{\Sigma}_{\mu\alpha} + b^2p^2\omega_{\mu\alpha}\right) + \theta_{\nu\alpha}\left(p^2\Lambda_{\mu\beta} + (b\cdot p)\,\tilde{\Sigma}_{\mu\beta} + b^2p^2\omega_{\mu\beta}\right) \\ + \left(b_\mu p_\beta - (b\cdot p)\,\omega_{\mu\beta}\right)\left(b_\alpha p_\nu - (b\cdot p)\,\omega_{\nu\alpha}\right) + \left(b_\mu p_\alpha - (b\cdot p)\,\omega_{\alpha\mu}\right)\left(b_\beta p_\nu - (b\cdot p)\,\omega_{\nu\beta}\right) \\ + \left(b_\nu p_\beta - (b\cdot p)\,\omega_{\nu\beta}\right)\left(b_\alpha p_\mu - (b\cdot p)\,\omega_{\mu\alpha}\right) + \left(b_\nu p_\alpha - (b\cdot p)\,\omega_{\alpha\nu}\right)\left(b_\beta p_\mu - (b\cdot p)\,\omega_{\mu\beta}\right)\Big],$$

desenvolvendo e agrupando, resulta

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big[p^2\left(\omega_{\mu\alpha}\Lambda_{\nu\beta} + \omega_{\mu\beta}\Lambda_{\nu\alpha} + \omega_{\nu\alpha}\Lambda_{\mu\beta} + \omega_{\nu\beta}\Lambda_{\mu\alpha}\right) \\ - (b\cdot p)\left(\omega_{\mu\alpha}\tilde{\Sigma}_{\nu\beta} + \omega_{\mu\beta}\tilde{\Sigma}_{\nu\alpha} + \omega_{\nu\alpha}\tilde{\Sigma}_{\mu\beta} + \omega_{\nu\beta}\tilde{\Sigma}_{\mu\alpha}\right) \\ + 4\,(b\cdot p)^2\,\omega_{\mu\nu}\omega_{\alpha\beta} + p^2\left(\theta_{\mu\alpha}\Lambda_{\nu\beta} + \theta_{\mu\beta}\Lambda_{\nu\alpha} + \theta_{\nu\alpha}\Lambda_{\mu\beta} + \theta_{\nu\beta}\Lambda_{\mu\alpha}\right) \\ + \left(\theta_{\mu\alpha}\tilde{\Sigma}_{\nu\beta} + \theta_{\mu\beta}\tilde{\Sigma}_{\nu\alpha} + \theta_{\nu\alpha}\tilde{\Sigma}_{\mu\beta} + \theta_{\nu\beta}\tilde{\Sigma}_{\mu\alpha}\right) \\ +b^2p^2\left(\omega_{\mu\alpha}\theta_{\nu\beta} + \omega_{\mu\beta}\theta_{\nu\alpha} + \omega_{\nu\alpha}\theta_{\mu\beta} + \omega_{\nu\beta}\theta_{\mu\alpha}\right)\Big],$$

identificamos

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - (b\cdot p)\,\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} + 2\,(b\cdot p)^2\,P^{(0-\omega)}_{\mu\nu,\alpha\beta} + p^2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + (b\cdot p)\,\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + b^2p^2P^{(1)}_{\mu\nu,\alpha\beta}. \tag{E.0.60}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right), \tag{E.0.61}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big[\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\left(\theta^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right) \\ + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\left(\theta^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right)\Big],$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big(\theta_{\mu\alpha}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\lambda}{}_{\beta} + \theta_{\mu\kappa}\Lambda^{\kappa}{}_{\beta}\tilde{\Sigma}_{\nu\lambda}\theta^{\lambda}{}_{\alpha} + \theta_{\mu\beta}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\lambda}{}_{\alpha} + \theta_{\mu\kappa}\Lambda^{\kappa}{}_{\alpha}\tilde{\Sigma}_{\nu\lambda}\theta^{\lambda}{}_{\beta}$$
$$+\theta_{\nu\alpha}\tilde{\Sigma}_{\mu\lambda}\Lambda^{\lambda}{}_{\beta} + \theta_{\nu\kappa}\Lambda^{\kappa}{}_{\beta}\tilde{\Sigma}_{\mu\lambda}\theta^{\lambda}{}_{\alpha} + \theta_{\nu\beta}\tilde{\Sigma}_{\mu\lambda}\Lambda^{\lambda}{}_{\alpha} + \theta_{\nu\kappa}\Lambda^{\kappa}{}_{\alpha}\tilde{\Sigma}_{\mu\lambda}\theta^{\lambda}{}_{\beta}\Big),$$

calculando a contração $\tilde{\Sigma}_{\nu\lambda}\Lambda^{\lambda}{}_{\beta}$,

$$\begin{aligned}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\lambda}{}_{\beta} &= (b_\nu p_\lambda + b_\lambda p_\nu)\, b^\lambda b_\beta,\\ &= (b\cdot p)\, b_\nu b_\beta + b^2 p_\nu b_\beta,\\ &= (b\cdot p)\,\Lambda_{\nu\beta} + b^2 b_\beta p_\nu,\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Bigg[&\theta_{\mu\alpha}\left((b\cdot p)\,\Lambda_{\nu\beta} + b^2 b_\beta p_\nu\right) + \left(\Lambda_{\mu\beta} - \frac{(b\cdot p)}{p^2} b_\beta p_\mu\right)(p_\nu b_\alpha - (b\cdot p)\,\omega_{\nu\alpha})\\ &+ \theta_{\mu\beta}\left((b\cdot p)\,\Lambda_{\nu\alpha} + b^2 b_\alpha p_\nu\right) + \left(\Lambda_{\mu\alpha} - \frac{(b\cdot p)}{p^2} b_\alpha p_\mu\right)(p_\nu b_\beta - (b\cdot p)\,\omega_{\nu\beta})\\ &+ \theta_{\nu\alpha}\left((b\cdot p)\,\Lambda_{\mu\beta} + b^2 b_\beta p_\mu\right) + \left(\Lambda_{\nu\beta} - \frac{(b\cdot p)}{p^2} b_\beta p_\nu\right)(p_\mu b_\alpha - (b\cdot p)\,\omega_{\mu\alpha})\\ &+\theta_{\nu\beta}\left((b\cdot p)\,\Lambda_{\mu\alpha} + b^2 b_\alpha p_\mu\right) + \left(\Lambda_{\nu\alpha} - \frac{(b\cdot p)}{p^2} b_\alpha p_\nu\right)(p_\mu b_\beta - (b\cdot p)\,\omega_{\mu\beta})\Bigg],\end{aligned}$$

desenvolvendo e agrupando os termos convenientemente, temos

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big[&(\theta_{\mu\alpha}\Lambda_{\nu\beta} + \theta_{\mu\beta}\Lambda_{\nu\alpha} + \theta_{\nu\alpha}\Lambda_{\mu\beta} + \theta_{\nu\beta}\Lambda_{\mu\alpha})\,(b\cdot p)\\ &+ b^2\,(\theta_{\mu\alpha} b_\beta p_\nu + \theta_{\mu\beta} b_\alpha p_\nu + \theta_{\nu\alpha} b_\beta p_\mu + \theta_{\nu\beta} b_\alpha p_\mu)\\ &+ (\Lambda_{\mu\alpha} b_\beta p_\nu + \Lambda_{\mu\beta} b_\alpha p_\nu + \Lambda_{\nu\alpha} b_\beta p_\mu + \Lambda_{\nu\beta} b_\alpha p_\mu)\\ &- (b\cdot p)\,(\omega_{\mu\alpha}\Lambda_{\nu\beta} + \omega_{\mu\beta}\Lambda_{\nu\alpha} + \omega_{\nu\alpha}\Lambda_{\mu\beta} + \omega_{\nu\beta}\Lambda_{\mu\alpha})\\ &-\frac{4\,(b\cdot p)}{p^2}\Lambda_{\alpha\beta}\omega_{\mu\nu}p^2 + \frac{2\,(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\alpha\beta}\omega_{\mu\nu}\Big],\end{aligned}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = (b\cdot p)\,\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + b^2\Pi^{(1-b)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{1}{2}(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}. \tag{E.0.62}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = (b\cdot p)\,\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + b^2\Pi^{(1-a)}_{\mu\nu,\alpha\beta} + \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{1}{2}(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)\,\Lambda_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \frac{\left(\theta_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \theta^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right)}{\sqrt{3}}, \tag{E.0.63}$$

$$\begin{aligned}&= \frac{1}{\sqrt{3}}\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\left(\theta_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \theta^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right) + \frac{1}{\sqrt{3}}\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\left(\theta_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \theta^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\ &= \frac{2}{\sqrt{3}}p^2\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)^2\,\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)\,\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\\ &= \frac{1}{\sqrt{3}}p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)^2\,\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}p^2\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)\,\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.\end{aligned} \tag{E.0.64}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)^2\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{2}{\sqrt{3}}p^2\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \frac{\left(\theta_{\alpha\beta}\Lambda^{\kappa\lambda} + \theta^{\kappa\lambda}\Lambda_{\alpha\beta}\right)}{\sqrt{3}}, \qquad \text{(E.0.65)}\\ &= \frac{1}{\sqrt{3}}\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\left(\theta_{\alpha\beta}\Lambda^{\kappa\lambda} + \theta^{\kappa\lambda}\Lambda_{\alpha\beta}\right) + \frac{1}{\sqrt{3}}\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\left(\theta_{\alpha\beta}\Lambda^{\kappa\lambda} + \theta^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\end{aligned}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\kappa\lambda}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\Lambda^{\kappa\lambda}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\mu\kappa}\theta^{\kappa\lambda}\tilde{\Sigma}_{\nu\lambda}\Lambda_{\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\nu\lambda}\theta^{\kappa\lambda}\tilde{\Sigma}_{\mu\kappa}\Lambda_{\alpha\beta},$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}b^2\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}(b\cdot p)\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)b^2\omega_{\mu\nu}\theta_{\alpha\beta}\\ &\quad - \frac{(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)\omega_{\mu\nu}\Lambda_{\alpha\beta}, \qquad \text{(E.0.66)}\end{aligned}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{b^2p^2 - (b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}(b\cdot p)\Lambda_{\mu\nu}\theta_{\alpha\beta}\\ &\quad - \frac{2}{\sqrt{3}}(b\cdot p)b^2\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)\omega_{\mu\nu}\Lambda_{\alpha\beta}. \qquad \text{(E.0.67)}\end{aligned}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{\sqrt{3}}(b\cdot p)\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)b^2\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2}{\sqrt{3}}(b\cdot p)\Lambda_{\mu\nu}\omega_{\alpha\beta}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \Lambda_{\alpha\beta}\Lambda^{\kappa\lambda}, \qquad \text{(E.0.68)}\\ &= \theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\kappa\lambda}\Lambda_{\alpha\beta} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\Lambda^{\kappa\lambda}\Lambda_{\alpha\beta},\end{aligned}$$

note que as contrações são análogas à primeira parte das anteriores,

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{b^2p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} + 2(b\cdot p)\Lambda_{\mu\nu}\Lambda_{\alpha\beta} - 2(b\cdot p)b^2\omega_{\mu\nu}\Lambda_{\alpha\beta},\\ &= \frac{b^2p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} + 2(b\cdot p)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2(b\cdot p)b^2\omega_{\mu\nu}\Lambda_{\alpha\beta}. \qquad \text{(E.0.69)}\end{aligned}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{b^2p^2 - (b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2(b\cdot p)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2(b\cdot p)b^2\Lambda_{\mu\nu}\omega_{\alpha\beta}.$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \left(\omega^\kappa{}_\alpha\Lambda^\lambda{}_\beta + \omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta + \omega^\kappa{}_\beta\Lambda^\lambda{}_\alpha + \omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha\right), \qquad \text{(E.0.70)}$$

$$\begin{aligned}\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\Big[&\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\left(\omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta + \omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha\right) + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa}\left(\omega^\kappa{}_\alpha\Lambda^\lambda{}_\beta + \omega^\kappa{}_\beta\Lambda^\lambda{}_\alpha\right)\\ &+\theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\left(\omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta + \omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha\right) + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\left(\omega^\kappa{}_\alpha\Lambda^\lambda{}_\beta + \omega^\kappa{}_\beta\Lambda^\lambda{}_\alpha\right)\Big],\end{aligned}$$

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = \theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\left(\omega^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta} + \omega^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right) + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\left(\omega^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta} + \omega^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha}\right),$$

podemos desenvolver os produtos e usar os resultados já calculados para escrever

$$\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} = 2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}. \tag{E.0.71}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = 2\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{2\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \left(\omega_{\alpha\beta}\Lambda^{\kappa\lambda} + \omega^{\kappa\lambda}\Lambda_{\alpha\beta}\right), \\
&= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\omega_{\alpha\beta}\Lambda^{\kappa\lambda}, \\
&= \theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\omega_{\alpha\beta}\Lambda^{\kappa\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\omega_{\alpha\beta}\Lambda^{\kappa\lambda}, \\
&= \left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\kappa\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\Lambda^{\kappa\lambda}\right)\omega_{\alpha\beta}, \\
&= b^2\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + 2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta} - 2\left(b\cdot p\right)b^2\omega_{\mu\nu}\omega_{\alpha\beta} - \frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}, \\
&= \frac{b^2p^2 - \left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + 2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta} - 2\left(b\cdot p\right)b^2P^{(0-\omega)}_{\mu\nu,\alpha\beta}. \qquad \text{(E.0.72)}
\end{aligned}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = \frac{b^2p^2 - \left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2\left(b\cdot p\right)\omega_{\mu\nu}\Lambda_{\alpha\beta} - 2\left(b\cdot p\right)b^2P^{(0-\omega)}_{\mu\nu,\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \left(\omega_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \omega^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right), \\
&= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right)\omega_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}, \\
&= \left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\tilde{\Sigma}^{\kappa\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\tilde{\Sigma}^{\kappa\lambda}\right)\omega_{\alpha\beta}, \\
&= 2p^2\Lambda_{\mu\nu}\omega_{\alpha\beta} - 2\left(b\cdot p\right)^2 P^{(0-\omega)}_{\mu\nu,\alpha\beta}. \qquad \text{(E.0.73)}
\end{aligned}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = 2p^2\omega_{\mu\nu}\Lambda\omega_{\alpha\beta} - 2\left(b\cdot p\right)^2 P^{(0-\omega)}_{\mu\nu,\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda} + \theta_{\mu\lambda}\tilde{\Sigma}_{\nu\kappa} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda} + \theta_{\nu\lambda}\tilde{\Sigma}_{\mu\kappa}\right) \times \left(\Lambda_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \Lambda^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right), \qquad \text{(E.0.74)} \\
&= \left(\theta_{\mu\kappa}\tilde{\Sigma}^{\kappa\lambda}\tilde{\Sigma}_{\nu\lambda} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\tilde{\Sigma}^{\kappa\lambda}\right)\Lambda_{\alpha\beta} + \left(\theta_{\mu\kappa}\tilde{\Sigma}_{\nu\lambda}\Lambda^{\kappa\lambda} + \theta_{\nu\kappa}\tilde{\Sigma}_{\mu\lambda}\Lambda^{\kappa\lambda}\right)\tilde{\Sigma}_{\alpha\beta},
\end{aligned}$$

$$\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} &= 2p^2\Lambda_{\mu\nu}\Lambda_{\alpha\beta} - 2\left(b\cdot p\right)^2\omega_{\mu\nu}\Lambda_{\alpha\beta} \\
&\quad + \left(\frac{b^2p^2 - \left(b\cdot p\right)^2}{p^2}\right)\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2\left(b\cdot p\right)\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\left(b\cdot p\right)b^2\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},
\end{aligned}$$

$$\begin{aligned}
\tilde{\Pi}^{(1)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} &= 2p^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2\left(b\cdot p\right)^2\omega_{\mu\nu}\Lambda_{\alpha\beta} + \left[b^2p^2 - \left(b\cdot p\right)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \\
&\quad + 2\left(b\cdot p\right)\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\left(b\cdot p\right)b^2\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(1)\kappa\lambda}_{,\alpha\beta} = 2p^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)^2\,\Lambda_{\mu\nu}\omega_{\alpha\beta} + \left[b^2p^2 - (b\cdot p)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + 2\,(b\cdot p)\,\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - 2\,(b\cdot p)\,b^2\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.$$

Para $\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\frac{1}{2}\left(\theta^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta} + \theta^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta} + \theta^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha} + \theta^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right), \tag{E.0.75}$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}\left(b^2\theta_{\mu\alpha}\Lambda_{\nu\beta} + b^2\theta_{\mu\beta}\Lambda_{\nu\alpha} + b^2\theta_{\nu\alpha}\Lambda_{\mu\beta} + b^2\theta_{\nu\beta}\Lambda_{\mu\alpha} + \theta_{\mu\kappa}\Lambda^{\kappa}{}_{\beta}\Lambda_{\nu\lambda}\theta^{\lambda}{}_{\alpha} + \theta_{\mu\kappa}\Lambda^{\kappa}{}_{\alpha}\Lambda_{\nu\lambda}\theta^{\lambda}{}_{\beta} + \theta_{\nu\kappa}\Lambda^{\kappa}{}_{\alpha}\Lambda_{\mu\lambda}\theta^{\lambda}{}_{\beta} + \theta_{\nu\kappa}\Lambda^{\kappa}{}_{\beta}\Lambda_{\mu\lambda}\theta^{\lambda}{}_{\alpha}\right),$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}b^2\left(\theta_{\mu\alpha}\Lambda_{\nu\beta} + \theta_{\mu\beta}\Lambda_{\nu\alpha} + \theta_{\nu\alpha}\Lambda_{\mu\beta} + \theta_{\nu\beta}\Lambda_{\mu\alpha}\right) + \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda^{\kappa}{}_{\beta}\Lambda_{\nu\lambda}\theta^{\lambda}{}_{\alpha} + \theta_{\mu\kappa}\Lambda^{\kappa}{}_{\alpha}\Lambda_{\nu\lambda}\theta^{\lambda}{}_{\beta} + \theta_{\nu\kappa}\Lambda^{\kappa}{}_{\alpha}\Lambda_{\mu\lambda}\theta^{\lambda}{}_{\beta} + \theta_{\nu\kappa}\Lambda^{\kappa}{}_{\beta}\Lambda_{\mu\lambda}\theta^{\lambda}{}_{\alpha}\right),$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{1}{2}b^2\left(\theta_{\mu\alpha}\Lambda_{\nu\beta} + \theta_{\mu\beta}\Lambda_{\nu\alpha} + \theta_{\nu\alpha}\Lambda_{\mu\beta} + \theta_{\nu\beta}\Lambda_{\mu\alpha}\right) + 2\Lambda_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\left(\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \Lambda_{\alpha\beta}\tilde{\Sigma}_{\mu\nu}\right) + \frac{(b\cdot p)^2}{2p^2}\left(\omega_{\mu\alpha}\Lambda_{\nu\beta} + \omega_{\mu\beta}\Lambda_{\nu\alpha} + \omega_{\nu\alpha}\Lambda_{\mu\beta} + \omega_{\nu\beta}\Lambda_{\mu\alpha}\right),$$

$$\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = b^2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + 2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\alpha\beta} + \frac{(b\cdot p)^2}{2p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}. \tag{E.0.76}$$

$$\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \theta^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\tilde{\Sigma}^{\kappa\lambda} + \theta_{\nu\kappa}\Lambda_{\mu\lambda}\tilde{\Sigma}^{\kappa\lambda}\right)\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\theta^{\kappa\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\theta^{\kappa\lambda}\right)\tilde{\Sigma}_{\alpha\beta},\\
&= \frac{2}{\sqrt{3}}\,(b\cdot p)\,\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\\
&= \frac{2}{\sqrt{3}}\,(b\cdot p)\,\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{1}{\sqrt{3}}\,(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.
\end{aligned} \tag{E.0.77}$$

$$\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\,(b\cdot p)\,\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{1}{\sqrt{3}}\,(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda} + \theta_{\mu\lambda}\Lambda_{\nu\kappa} + \theta_{\nu\kappa}\Lambda_{\mu\lambda} + \theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\Lambda^{\kappa\lambda} + \theta^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\Lambda^{\kappa\lambda} + \theta_{\nu\kappa}\Lambda_{\mu\lambda}\Lambda^{\kappa\lambda}\right)\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\theta^{\kappa\lambda} + \theta_{\nu\kappa}\Lambda_{\mu\lambda}\theta^{\kappa\lambda}\right)\Lambda_{\alpha\beta},\\
&= \frac{2}{\sqrt{3}}b^2\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\Lambda_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta},\\
&= \frac{2}{\sqrt{3}}b^2\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{\sqrt{3}}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}.
\end{aligned} \tag{E.0.78}$$

$$\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{\sqrt{3}}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{\sqrt{3}p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\Lambda_{\alpha\beta}\Lambda^{\kappa\lambda},\\
&= \left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\Lambda^{\kappa\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\Lambda^{\kappa\lambda}\right)\Lambda_{\alpha\beta},\\
&= 2b^2\Lambda_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta},\\
&= 2b^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)\,b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}.
\end{aligned} \tag{E.0.79}$$

$$\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2b^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)\,b^2}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\left(\omega^\kappa{}_\alpha\Lambda^\lambda{}_\beta+\omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta+\omega^\kappa{}_\beta\Lambda^\lambda{}_\alpha+\omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha\right),\\
&= \frac{1}{2}\left(\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\mu\kappa}\Lambda_{\nu\lambda}\right)\left(\omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta+\omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha\right)+\frac{1}{2}\left(\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\left(\omega^\kappa{}_\alpha\Lambda^\lambda{}_\beta+\omega^\kappa{}_\beta\Lambda^\lambda{}_\alpha\right)\\
&= \theta_{\mu\kappa}\Lambda_{\nu\lambda}\omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta+\theta_{\mu\kappa}\Lambda_{\nu\lambda}\omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha+\theta_{\nu\kappa}\Lambda_{\mu\lambda}\omega^\lambda{}_\alpha\Lambda^\kappa{}_\beta+\theta_{\nu\kappa}\Lambda_{\mu\lambda}\omega^\lambda{}_\beta\Lambda^\kappa{}_\alpha,\\
&= \frac{2\,(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\left(\omega_{\mu\alpha}\Lambda_{\nu\beta}+\omega_{\mu\beta}\Lambda_{\nu\alpha}+\omega_{\nu\alpha}\Lambda_{\mu\beta}+\omega_{\nu\beta}\Lambda_{\mu\alpha}\right),\\
&= \frac{2\,(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.
\end{aligned} \tag{E.0.80}$$

$$\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = \frac{2\,(b\cdot p)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\left(\omega_{\alpha\beta}\Lambda^{\kappa\lambda}+\omega^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\\
&= \left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\Lambda^{\kappa\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\Lambda^{\kappa\lambda}\right)\omega_{\alpha\beta},\\
&= 2b^2\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned} \tag{E.0.81}$$

$$\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2b^2\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\left(\omega_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\omega^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \left(\tilde{\Sigma}^{\kappa\lambda}\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\tilde{\Sigma}^{\kappa\lambda}\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\omega_{\alpha\beta},\\
&= 2\,(b\cdot p)\,\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned} \tag{E.0.82}$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.$$

$$
\begin{aligned}
\tilde{\Pi}^{(2)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{2}\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}+\theta_{\mu\lambda}\Lambda_{\nu\kappa}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}+\theta_{\nu\lambda}\Lambda_{\mu\kappa}\right)\times\left(\Lambda_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\Lambda^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\tilde{\Sigma}^{\kappa\lambda}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}\tilde{\Sigma}^{\kappa\lambda}\right)\Lambda_{\alpha\beta}+\left(\theta_{\mu\kappa}\Lambda_{\nu\lambda}\Lambda^{\kappa\lambda}+\theta_{\nu\kappa}\Lambda_{\mu\lambda}\Lambda^{\kappa\lambda}\right)\tilde{\Sigma}_{\alpha\beta},\\
&= 2\left(b\cdot p\right)\Lambda_{\mu\nu}\Lambda_{\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}+2b^2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{\left(b\cdot p\right)b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\\
&= 2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}+2b^2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\left(b\cdot p\right)b^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}. \qquad \text{(E.0.83)}
\end{aligned}
$$

$$
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(2)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+2b^2\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}-\left(b\cdot p\right)b^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.
$$

Para $\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta}$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\theta^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \frac{1}{3}\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}\theta_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\frac{1}{3}\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\theta^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta},\\
&= \frac{2}{3}\left(b^2p^2+\left(b\cdot p\right)^2\right)\theta_{\mu\nu}\theta_{\alpha\beta}+\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\\
&= 2\left[b^2p^2+\left(b\cdot p\right)^2\right]P^{(0-\theta)}_{\mu\nu,\alpha\beta}+p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}. \qquad \text{(E.0.84)}
\end{aligned}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\Lambda^{\kappa\lambda}+\theta^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\\
&= \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}\theta_{\kappa\lambda}\Lambda^{\kappa\lambda}+\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta},\\
&= \frac{2}{3}\left(b\cdot p\right)b^2\theta_{\mu\nu}\theta_{\alpha\beta}+\frac{\left(b^2p^2-\left(b\cdot p\right)^2\right)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta},\\
&= 2\left(b\cdot p\right)b^2P^{(0-\theta)}_{\mu\nu,\alpha\beta}+\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}. \qquad \text{(E.0.85)}
\end{aligned}
$$

$$
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = 2\left(b\cdot p\right)b^2P^{(0-\theta)}_{\mu\nu,\alpha\beta}+\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\Lambda_{\alpha\beta}\Lambda^{\kappa\lambda},\\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\theta_{\kappa\lambda}\Lambda^{\kappa\lambda},\\
&= \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}. \qquad \text{(E.0.86)}
\end{aligned}
$$

$$
\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\left(\omega^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta}+\omega^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta}+\omega^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha}+\omega^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right),\\
&= \frac{2}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}\left(\omega^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta}+\omega^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right),\\
&= \frac{2\left[b^2p^2+(b\cdot p)^2\right]}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned}
\tag{E.0.87}
$$

$$
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2\left[b^2p^2+(b\cdot p)^2\right]}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\left(\omega_{\alpha\beta}\Lambda^{\kappa\lambda}+\omega^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\omega^{\kappa\lambda}+\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\theta_{\kappa\lambda}\Lambda^{\kappa\lambda},\\
&= \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned}
\tag{E.0.88}
$$

$$
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}(b\cdot p)\,b^2\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\left(\omega_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\omega^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\tilde{\Sigma}^{\kappa\lambda}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\omega^{\kappa\lambda},\\
&= \frac{2}{\sqrt{3}}\left[b^2p^2+(b\cdot p)^2\right]\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned}
\tag{E.0.89}
$$

$$
\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left[b^2p^2+(b\cdot p)^2\right]\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\tilde{\Sigma}_{\kappa\lambda}+\theta_{\kappa\lambda}\tilde{\Sigma}_{\mu\nu}\right)\times\left(\Lambda_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\Lambda^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\tilde{\Sigma}^{\kappa\lambda}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\tilde{\Sigma}_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{\sqrt{3}}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\Lambda^{\kappa\lambda}\theta_{\kappa\lambda},\\
&= \frac{2}{\sqrt{3}}\left[b^2p^2+(b\cdot p)^2\right]\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,b^2\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{\left[b^2p^2-(b\cdot p)^2\right]}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},\\
&= \frac{2}{\sqrt{3}}\left[b^2p^2+(b\cdot p)^2\right]\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,b^2\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\frac{1}{\sqrt{3}}\left[b^2p^2-(b\cdot p)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.
\end{aligned}
\tag{E.0.90}
$$

$$
\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Sigma)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left[b^2p^2+(b\cdot p)^2\right]\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{2}{\sqrt{3}}(b\cdot p)\,b^2\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\frac{1}{\sqrt{3}}\left[b^2p^2-(b\cdot p)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}.
$$

Para $\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta}$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\frac{1}{\sqrt{3}}\left(\theta_{\alpha\beta}\Lambda^{\kappa\lambda}+\theta^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\\
&= \frac{1}{3}\theta_{\mu\nu}\theta_{\alpha\beta}\Lambda_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{3}\theta_{\mu\nu}\Lambda_{\alpha\beta}\Lambda_{\kappa\lambda}\theta^{\kappa\lambda}+\frac{1}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta}\theta_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{3}\Lambda_{\mu\nu}\Lambda_{\alpha\beta}\theta_{\kappa\lambda}\theta^{\kappa\lambda}\\
&= \frac{1}{3}b^4\theta_{\mu\nu}\theta_{\alpha\beta}+\frac{\left(b^2p^2-(b\cdot p)^2\right)}{3p^2}\left(\theta_{\mu\nu}\Lambda_{\alpha\beta}+\Lambda_{\mu\nu}\theta_{\alpha\beta}\right)+\Lambda_{\mu\nu}\Lambda_{\alpha\beta},\\
&= b^4P^{(0-\theta)}_{\mu\nu,\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta}+\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}.
\end{aligned} \tag{E.0.91}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Lambda)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\Lambda_{\alpha\beta}\Lambda^{\kappa\lambda},\\
&= \frac{1}{\sqrt{3}}\Lambda_{\alpha\beta}\theta_{\mu\nu}\Lambda_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{\sqrt{3}}\Lambda_{\alpha\beta}\Lambda_{\mu\nu}\Lambda^{\kappa\lambda}\theta_{\kappa\lambda},\\
&= \frac{1}{\sqrt{3}}b^4\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}.
\end{aligned} \tag{E.0.92}
$$

$$
\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b^4\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-a)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\left(\omega^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta}+\omega^{\lambda}{}_{\alpha}\Lambda^{\kappa}{}_{\beta}+\omega^{\kappa}{}_{\beta}\Lambda^{\lambda}{}_{\alpha}+\omega^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right),\\
&= \frac{2}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}\omega^{\kappa}{}_{\alpha}\Lambda^{\lambda}{}_{\beta}+\theta_{\mu\nu}\Lambda_{\kappa\lambda}\omega^{\lambda}{}_{\beta}\Lambda^{\kappa}{}_{\alpha}\right),\\
&= \frac{2\,(b\cdot p)\,b^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned} \tag{E.0.93}
$$

$$
\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2\,(b\cdot p)\,b^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Lambda-b)\kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\left(\omega_{\alpha\beta}\Lambda^{\kappa\lambda}+\omega^{\kappa\lambda}\Lambda_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}\Lambda_{\kappa\lambda}\Lambda^{\kappa\lambda}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}\Lambda_{\kappa\lambda}\omega^{\kappa\lambda}+\frac{1}{\sqrt{3}}\Lambda_{\mu\nu}\omega_{\alpha\beta}\theta_{\kappa\lambda}\Lambda^{\kappa\lambda},\\
&= \frac{1}{\sqrt{3}}b^4\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned} \tag{E.0.94}
$$

$$
\tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{1}{\sqrt{3}}b^4\omega_{\mu\nu}\theta_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta}.
$$

$$
\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\omega\Sigma)\ \kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda}+\theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\times\left(\omega_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda}+\omega^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right),\\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\omega_{\alpha\beta}\Lambda_{\kappa\lambda}\tilde{\Sigma}^{\kappa\lambda}+\frac{1}{\sqrt{3}}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\Lambda_{\kappa\lambda}\omega^{\kappa\lambda},\\
&= \frac{2}{\sqrt{3}}\,(b\cdot p)\,b^2\theta_{\mu\nu}\omega_{\alpha\beta}+\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned} \tag{E.0.95}
$$

$$\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}.$$

$$\begin{aligned}
\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\Lambda\Sigma)\ \kappa\lambda}_{,\alpha\beta} &= \frac{1}{\sqrt{3}}\left(\theta_{\mu\nu}\Lambda_{\kappa\lambda} + \theta_{\kappa\lambda}\Lambda_{\mu\nu}\right)\left(\Lambda_{\alpha\beta}\tilde{\Sigma}^{\kappa\lambda} + \Lambda^{\kappa\lambda}\tilde{\Sigma}_{\alpha\beta}\right), \\
&= \frac{1}{\sqrt{3}}\theta_{\mu\nu}\Lambda_{\alpha\beta}\Lambda_{\kappa\lambda}\tilde{\Sigma}^{\kappa\lambda} + \frac{1}{\sqrt{3}}\Lambda_{\kappa\lambda}\Lambda^{\kappa\lambda}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{1}{\sqrt{3}}\theta_{\kappa\lambda}\Lambda^{\kappa\lambda}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}, \\
&= \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{1}{\sqrt{3}}b^4\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}. \qquad \text{(E.0.96)}
\end{aligned}$$

$$\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\kappa\lambda}\tilde{\Pi}^{(\theta\Lambda)\kappa\lambda}_{,\alpha\beta} = \frac{2}{\sqrt{3}}\left(b\cdot p\right)b^2\Lambda_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}}b^4\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}.$$

APÊNDICE F

Cálculo do propagador com violação da simetria de Lorentz

Aqui apresentaremos os passos omitidos no cálculo do propagador

$$\mathcal{O}_{\mu\nu,\alpha\beta} = a_1 P^{(1)}_{\mu\nu,\alpha\beta} + a_2 P^{(2)}_{\mu\nu,\alpha\beta} + a_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta} + a_4 P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} + a_5 \tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + a_6 \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + a_7 \tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} + a_8 \tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta}, \tag{F.0.1}$$

$$\begin{aligned}\mathcal{O}^{-1}_{\mu\nu,\alpha\beta} &= b_1 P^{(1)}_{\mu\nu,\alpha\beta} + b_2 P^{(2)}_{\mu\nu,\alpha\beta} + b_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta} + b_4 P^{(0-\omega)}_{\mu\nu,\alpha\beta} + b_5 P^{(0-\theta\omega)}_{\mu\nu,\alpha\beta} + b_6 \tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + b_7 \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} \\ &+ b_8 \tilde{\Pi}^{(\theta\Sigma)}_{\mu\nu,\alpha\beta} + b_9 \tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} + b_{10} \tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} + b_{11} \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + b_{12} \tilde{\Pi}^{(\omega\Lambda-b)}_{\mu\nu,\alpha\beta} + b_{13} \tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} + b_{14} \tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\alpha\beta}.\end{aligned}$$

Façamos $\mathcal{O}\mathcal{O}^{-1} = \mathcal{I}$,

$$\begin{aligned}a_1 P^{(1)} \mathcal{O}^{-1} &= a_1 b_1 P^{(1)}_{\mu\nu,\alpha\beta} + a_1 b_6 \left[(b\cdot p)\, P^{(1)}_{\mu\nu,\alpha\beta} + \Pi^{(1-b)}_{\mu\nu,\alpha\beta} \right] + a_1 b_7 \frac{(b\cdot p)}{p^2} \Pi^{(1-b)}_{\mu\nu,\alpha\beta} \\ &+ \frac{1}{\sqrt{3}} a_1 b_8 \left[\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta} \right] + \frac{1}{\sqrt{3}} a_1 b_9 \left[\frac{(b\cdot p)}{p^2} \tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} - 2 \frac{(b\cdot p)^2}{p^2} \omega_{\mu\nu}\theta_{\alpha\beta} \right] \\ &+ a_1 b_{10} \left[\frac{(b\cdot p)}{p^2} \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - 2 \frac{(b\cdot p)^2}{p^2} \omega_{\mu\nu}\Lambda_{\alpha\beta} \right] + a_1 b_{11} \left[\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2 \frac{(b\cdot p)}{p^2} \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} \right] \\ &+ a_1 b_{12} \left[\frac{(b\cdot p)}{p^2} \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - \frac{2\,(b\cdot p)^2}{p^2} P^{(0-\omega)}_{\mu\nu,\alpha\beta} \right] + a_1 b_{13} \left[\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - 2\,(b\cdot p)\, P^{(0-\omega)}_{\mu\nu,\alpha\beta} \right] \\ &+ a_1 b_{14} \left[\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta} + (b\cdot p)\, \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} - \frac{2\,(b\cdot p)^2}{p^2} \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} \right],\end{aligned}$$

$$
\begin{aligned}
a_1 P^{(1)}\mathcal{O}^{-1} = {} & \left[a_1 b_1 + a_1 b_6 \left(b\cdot p\right)\right] P^{(1)}_{\mu\nu,\alpha\beta} - \left[2a_1 b_{13}\left(b\cdot p\right) + 2a_1 b_{12}\frac{\left(b\cdot p\right)^2}{p^2}\right] P^{(0-\omega)}_{\mu\nu,\alpha\beta} \\
& + \left[a_1 b_6 + a_1 b_7 \frac{\left(b\cdot p\right)}{p^2}\right] \Pi^{(1-b)}_{\mu\nu,\alpha\beta} + \left[a_1 b_{14}\left(b\cdot p\right) + a_1 b_{11}\right] \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \\
& + \left[\frac{1}{\sqrt{3}} a_1 b_8 + a_1 b_9 \frac{\left(b\cdot p\right)}{\sqrt{3}p^2}\right] \tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} - \left[\frac{2}{\sqrt{3}} a_1 b_8 \left(b\cdot p\right) + \frac{2}{\sqrt{3}} a_1 b_9 \frac{\left(b\cdot p\right)^2}{p^2}\right] \omega_{\mu\nu}\theta_{\alpha\beta} \\
& + \left[a_1 b_{12}\frac{\left(b\cdot p\right)}{p^2} + a_1 b_{13}\right] \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + \left[a_1 b_{14} + a_1 b_{10}\frac{\left(b\cdot p\right)}{p^2}\right] \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} \\
& - \left[2a_1 b_{11}\frac{\left(b\cdot p\right)}{p^2} + 2a_1 b_{14}\frac{\left(b\cdot p\right)^2}{p^2}\right] \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \left[2a_1 b_{10}\frac{\left(b\cdot p\right)^2}{p^2} + 2a_1 b_{14}\left(b\cdot p\right)\right] \omega_{\mu\nu}\Lambda_{\alpha\beta}.
\end{aligned}
$$

$$
\begin{aligned}
a_2 P^{(2)}\mathcal{O}^{-1} = {} & a_2 b_2 P^{(2)}_{\mu\nu,\alpha\beta} + a_2 b_6 \left[\Pi^{(1-a)}_{\mu\nu,\alpha\beta} - \left(b\cdot p\right) P^{(1)}_{\mu\nu,\alpha\beta} - \frac{1}{3}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{2}{3}\left(b\cdot p\right)\theta_{\mu\nu}\omega_{\alpha\beta}\right] \\
& + a_2 b_7 \left[\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} - \frac{\left(b\cdot p\right)}{p^2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta} - \frac{2}{3}\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{\left(b\cdot p\right)}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right] \\
& + \frac{1}{\sqrt{3}} a_2 b_9 \left[\Lambda_{\mu\nu}\theta_{\alpha\beta} - \frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\theta_{\alpha\beta} - \frac{b^2p^2 - \left(b\cdot p\right)^2}{p^2} P^{(0-\theta)}_{\mu\nu,\alpha\beta}\right] \\
& + a_2 b_{10} \left[\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} + \frac{\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta} - \frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta}\right] \\
& + a_2 b_{12} \left[\Lambda_{\mu\nu}\omega_{\alpha\beta} - \frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + \frac{\left(b\cdot p\right)^2}{p^2} P^{(0-\omega)}_{\mu\nu,\alpha\beta} - \frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta}\right] \\
& + a_2 b_{14} \left[\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} + \frac{\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right],
\end{aligned}
$$

$$
\begin{aligned}
a_2 P^{(2)}\mathcal{O}^{-1} = {} & -a_2 b_6 \left(b\cdot p\right) P^{(1)}_{\mu\nu,\alpha\beta} + a_2 b_2 P^{(2)}_{\mu\nu,\alpha\beta} - a_2 b_9 \frac{b^2p^2 - \left(b\cdot p\right)^2}{\sqrt{3}p^2} P^{(0-\theta)}_{\mu\nu,\alpha\beta} + a_2 b_{12}\frac{\left(b\cdot p\right)^2}{p^2} P^{(0-\omega)}_{\mu\nu,\alpha\beta} \\
& + a_2 b_7 \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} - a_2 b_7 \frac{\left(b\cdot p\right)}{p^2}\Pi^{(1-b)}_{\mu\nu,\alpha\beta} + a_2 b_6 \Pi^{(1-a)}_{\mu\nu,\alpha\beta} - a_2 b_{14}\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} + a_2 b_{10}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} \\
& + a_2 b_9 \frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}\omega_{\mu\nu}\theta_{\alpha\beta} + \frac{1}{\sqrt{3}} a_2 b_9 \Lambda_{\mu\nu}\theta_{\alpha\beta} - a_2 b_9 \frac{\left(b\cdot p\right)}{\sqrt{3}p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} \\
& + a_2 b_{12}\Lambda_{\mu\nu}\omega_{\alpha\beta} - a_2 b_{12}\frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + \left[\frac{2}{3} a_2 b_6 \left(b\cdot p\right) - a_2 b_{12}\frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\right] \theta_{\mu\nu}\omega_{\alpha\beta} \\
& + a_2 b_{10}\frac{\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\Lambda_{\alpha\beta} - a_2 b_{10}\frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \left[a_2 b_{10}\frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2} + \frac{2}{3} a_2 b_7\right] \theta_{\mu\nu}\Lambda_{\alpha\beta} \\
& + \left[a_2 b_7 \frac{\left(b\cdot p\right)}{3p^2} - \frac{1}{3} a_2 b_6 - a_2 b_{14}\frac{b^2p^2 - \left(b\cdot p\right)^2}{3p^2}\right] \theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} \\
& + a_2 b_{14}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + a_2 b_{14}\frac{\left(b\cdot p\right)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
a_3 P^{(0-\theta)} \mathcal{O}^{-1} = {} & a_3 b_3 P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{1}{\sqrt{3}} a_3 b_5 \theta_{\mu\nu} \omega_{\alpha\beta} + a_3 b_6 \left[\frac{1}{3} \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta} - \frac{2}{3} \left(b \cdot p\right) \theta_{\mu\nu} \omega_{\alpha\beta}\right] \\
& + a_3 b_7 \left[\frac{2}{3} \theta_{\mu\nu} \Lambda_{\alpha\beta} - \frac{(b \cdot p)}{3p^2} \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta}\right] + \frac{1}{\sqrt{3}} a_3 b_8 \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta} \\
& + \frac{1}{\sqrt{3}} a_3 b_9 \left[\frac{b^2 p^2 - (b \cdot p)^2}{p^2} P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \theta_{\mu\nu} \Lambda_{\alpha\beta}\right] + a_3 b_{10} \frac{b^2 p^2 - (b \cdot p)^2}{3p^2} \theta_{\mu\nu} \Lambda_{\alpha\beta} \\
& + a_3 b_{12} \frac{b^2 p^2 - (b \cdot p)^2}{3p^2} \theta_{\mu\nu} \omega_{\alpha\beta} + a_3 b_{14} \frac{b^2 p^2 - (b \cdot p)^2}{3p^2} \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta},
\end{aligned}
$$

$$
\begin{aligned}
a_3 P^{(0-\theta)} \mathcal{O}^{-1} = {} & \left[a_3 b_3 + a_3 b_9 \frac{b^2 p^2 - (b \cdot p)^2}{\sqrt{3} p^2}\right] P^{(0-\theta)}_{\mu\nu,\alpha\beta} \\
& + \left[a_3 b_{12} \frac{b^2 p^2 - (b \cdot p)^2}{3p^2} + \frac{1}{\sqrt{3}} a_3 b_5 - \frac{2}{3} a_3 b_6 \left(b \cdot p\right)\right] \theta_{\mu\nu} \omega_{\alpha\beta} \\
& + \left[\frac{1}{\sqrt{3}} a_3 b_8 + \frac{1}{3} a_3 b_6 + a_3 b_{14} \frac{b^2 p^2 - (b \cdot p)^2}{3p^2} - a_3 b_7 \frac{(b \cdot p)}{3p^2}\right] \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta} \\
& + \left[\frac{1}{\sqrt{3}} a_3 b_9 + \frac{2}{3} a_3 b_7 + a_3 b_{10} \frac{b^2 p^2 - (b \cdot p)^2}{3p^2}\right] \theta_{\mu\nu} \Lambda_{\alpha\beta}.
\end{aligned}
$$

$$
\begin{aligned}
a_4 P^{(0-\theta\omega)} \mathcal{O}^{-1} = {} & \frac{1}{\sqrt{3}} a_4 b_3 \theta_{\mu\nu} \omega_{\alpha\beta} + \frac{1}{\sqrt{3}} a_4 b_4 \omega_{\mu\nu} \theta_{\alpha\beta} + a_4 b_5 \left(P^{(0-\theta)}_{\mu\nu,\alpha\beta} + P^{(0-\omega)}_{\mu\nu,\alpha\beta}\right) \\
& + \frac{1}{\sqrt{3}} a_4 b_6 \left[\omega_{\mu\nu} \tilde{\Sigma}_{\alpha\beta} - 2 \left(b \cdot p\right) P^{(0-\omega)}_{\mu\nu,\alpha\beta}\right] + \frac{1}{\sqrt{3}} a_4 b_7 \left[2 \omega_{\mu\nu} \Lambda_{\alpha\beta} - \frac{(b \cdot p)}{p^2} \omega_{\mu\nu} \tilde{\Sigma}_{\alpha\beta}\right] \\
& + a_4 b_8 \left[2 \left(b \cdot p\right) P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \omega_{\mu\nu} \tilde{\Sigma}_{\alpha\beta}\right] + a_4 b_9 \left[\frac{(b \cdot p)^2}{p^2} P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2 p^2 - (b \cdot p)^2}{3p^2} \omega_{\mu\nu} \theta_{\alpha\beta} + \omega_{\mu\nu} \Lambda_{\alpha\beta}\right] \\
& + \frac{1}{\sqrt{3}} a_4 b_{10} \left[\frac{(b \cdot p)^2}{p^2} \theta_{\mu\nu} \Lambda_{\alpha\beta} + \frac{b^2 p^2 - (b \cdot p)^2}{p^2} \omega_{\mu\nu} \Lambda_{\alpha\beta}\right] + \frac{2}{\sqrt{3}} a_4 b_{11} \frac{(b \cdot p)}{p^2} \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta} \\
& + \frac{1}{\sqrt{3}} a_4 b_{12} \left[\frac{(b \cdot p)^2}{p^2} \theta_{\mu\nu} \omega_{\alpha\beta} + \frac{b^2 p^2 - (b \cdot p)^2}{p^2} P^{(0-\omega)}_{\mu\nu,\alpha\beta} + \theta_{\mu\nu} \Lambda_{\alpha\beta}\right] + \frac{1}{\sqrt{3}} a_4 b_{13} \left[2 \left(b \cdot p\right) \theta_{\mu\nu} \omega_{\alpha\beta} + \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta}\right] \\
& + \frac{1}{\sqrt{3}} a_4 b_{14} \left[2 \left(b \cdot p\right) \theta_{\mu\nu} \Lambda_{\alpha\beta} + \frac{(b \cdot p)^2}{p^2} \theta_{\mu\nu} \tilde{\Sigma}_{\alpha\beta} + \frac{b^2 p^2 - (b \cdot p)^2}{p^2} \omega_{\mu\nu} \tilde{\Sigma}_{\alpha\beta}\right].
\end{aligned}
$$

$$a_4 P^{(0-\theta\omega)}\mathcal{O}^{-1} = \left[a_4 b_5 + 2a_4 b_8\,(b\cdot p) + a_4 b_9 \frac{(b\cdot p)^2}{p^2}\right] P^{(0-\theta)}_{\mu\nu,\alpha\beta}$$

$$+ \left[a_4 b_5 - \frac{2}{\sqrt{3}} a_4 b_6\,(b\cdot p) + a_4 b_{12} \frac{b^2 p^2 - (b\cdot p)^2}{\sqrt{3} p^2}\right] P^{(0-\omega)}_{\mu\nu,\alpha\beta}$$

$$+ \left[\frac{1}{\sqrt{3}} a_4 b_4 + a_4 b_9 \frac{b^2 p^2 - (b\cdot p)^2}{3p^2}\right] \omega_{\mu\nu}\theta_{\alpha\beta} + \left[\frac{1}{\sqrt{3}} a_4 b_3 + \frac{2}{\sqrt{3}} a_4 b_{13}\,(b\cdot p) + a_4 b_{12} \frac{(b\cdot p)^2}{\sqrt{3} p^2}\right] \theta_{\mu\nu}\omega_{\alpha\beta}$$

$$+ \left[\frac{1}{\sqrt{3}} a_4 b_{13} + \frac{2}{\sqrt{3}} a_4 b_{11} \frac{(b\cdot p)}{p^2} + a_4 b_{14} \frac{(b\cdot p)^2}{\sqrt{3} p^2}\right] \theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}$$

$$+ \left[a_4 b_8 + \frac{1}{\sqrt{3}} a_4 b_6 - a_4 b_7 \frac{(b\cdot p)}{\sqrt{3} p^2} + a_4 b_{14} \frac{b^2 p^2 - (b\cdot p)^2}{\sqrt{3} p^2}\right] \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}$$

$$+ \left[a_4 b_9 + \frac{2}{\sqrt{3}} a_4 b_7 + a_4 b_{10} \frac{b^2 p^2 - (b\cdot p)^2}{\sqrt{3} p^2}\right] \omega_{\mu\nu}\Lambda_{\alpha\beta}$$

$$+ \left[\frac{1}{\sqrt{3}} a_4 b_{12} + \frac{2}{\sqrt{3}} a_4 b_{14}\,(b\cdot p) + a_4 b_{10} \frac{(b\cdot p)^2}{\sqrt{3} p^2}\right] \theta_{\mu\nu}\Lambda_{\alpha\beta}.$$

$$a_5 \tilde{\Pi}^{(1)}\mathcal{O}^{-1} = a_5 b_1 \left[(b\cdot p)\, P^{(1)}_{\mu\nu,\alpha\beta} + \Pi^{(1-a)}_{\mu\nu,\alpha\beta}\right] + a_5 b_2 \left[\Pi^{(1-b)}_{\mu\nu,\alpha\beta} - (b\cdot p)\, P^{(1)}_{\mu\nu,\alpha\beta} - \frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \frac{2}{3}(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta}\right]$$

$$+ a_5 b_3 \left[\frac{1}{3}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} - \frac{2}{3}(b\cdot p)\,\omega_{\mu\nu}\theta_{\alpha\beta}\right] + \frac{1}{\sqrt{3}} a_5 b_5 \left[\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} - 2\,(b\cdot p)\, P^{(0-\omega)}_{\mu\nu,\alpha\beta}\right]$$

$$+ a_5 b_6 \left[\frac{1}{2} p^2 \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - (b\cdot p)\,\tilde{\Pi}^{(\omega\Sigma)}_{\mu\nu,\alpha\beta} + 2\,(b\cdot p)^2 P^{(0-\omega)}_{\mu\nu,\alpha\beta} + p^2 \tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + (b\cdot p)\,\tilde{\Pi}^{(1)}_{\mu\nu,\alpha\beta} + b^2 p^2 P^{(1)}_{\mu\nu,\alpha\beta}\right]$$

$$+ a_5 b_7 \left[(b\cdot p)\,\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + b^2 \Pi^{(1-b)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \frac{1}{2}(b\cdot p)\,\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right]$$

$$+ \frac{1}{\sqrt{3}} a_5 b_8 \left[p^2 \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)^2 \omega_{\mu\nu}\theta_{\alpha\beta} + 2p^2 \Lambda_{\mu\nu}\theta_{\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right]$$

$$+ \frac{1}{\sqrt{3}} a_5 b_9 \left[\frac{b^2 p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + 2\,(b\cdot p)\,\Lambda_{\mu\nu}\theta_{\alpha\beta} - 2\,(b\cdot p)\, b^2 \omega_{\mu\nu}\theta_{\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - 2\,(b\cdot p)\,\omega_{\mu\nu}\Lambda_{\alpha\beta}\right]$$

$$+ a_5 b_{10} \left[\frac{b^2 p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} + 2\,(b\cdot p)\,\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)\, b^2 \omega_{\mu\nu}\Lambda_{\alpha\beta}\right]$$

$$+ a_5 b_{11} \left[2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - \frac{2\,(b\cdot p)^2}{p^2}\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right] + a_5 b_{13} \left[2p^2 \Lambda_{\mu\nu}\omega_{\alpha\beta} - 2\,(b\cdot p)^2 P^{(0-\omega)}_{\mu\nu,\alpha\beta}\right]$$

$$+ a_5 b_{12} \left[\frac{b^2 p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} + 2\,(b\cdot p)\,\Lambda_{\mu\nu}\omega_{\alpha\beta} - 2\,(b\cdot p)\, b^2 P^{(0-\omega)}_{\mu\nu,\alpha\beta}\right]$$

$$+ a_5 b_{14} \left[2p^2 \tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - 2\,(b\cdot p)^2 \omega_{\mu\nu}\Lambda_{\alpha\beta} + \left[b^2 p^2 - (b\cdot p)^2\right] \tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda} + 2\,(b\cdot p)\,\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\,(b\cdot p)\, b^2 \omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right],$$

$$
\begin{aligned}
a_5\tilde{\Pi}^{(1)}\mathcal{O}^{-1} = &\left[a_5b_1\left(b\cdot p\right)+a_5b_6b^2p^2-a_5b_2\left(b\cdot p\right)\right]P^{(1)}_{\mu\nu,\alpha\beta}\\
&+\left[2a_5b_6\left(b\cdot p\right)^2-\frac{2}{\sqrt{3}}a_5b_5\left(b\cdot p\right)-2a_5b_{13}\left(b\cdot p\right)^2-2a_5b_{12}\left(b\cdot p\right)b^2\right]P^{(0-\omega)}_{\mu\nu,\alpha\beta}\\
&+\left[a_5b_6p^2+a_5b_7\left(b\cdot p\right)\right]\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+\left[a_5b_1+a_5b_6\left(b\cdot p\right)\right]\Pi^{(1-a)}_{\mu\nu,\alpha\beta}+\left[a_5b_2+a_5b_6\left(b\cdot p\right)+a_5b_7b^2\right]\Pi^{(1-b)}_{\mu\nu,\alpha\beta}\\
&+\left\{\frac{1}{\sqrt{3}}a_5b_8p^2+\frac{1}{2}a_5b_6p^2-\frac{1}{2}a_5b_7\left(b\cdot p\right)+a_5b_{14}\left[b^2p^2-\left(b\cdot p\right)^2\right]\right\}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\kappa\lambda}\\
&+\left[2a_5b_{10}\left(b\cdot p\right)+2a_5b_{14}p^2\right]\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}\\
&-\left[2a_5b_{10}\left(b\cdot p\right)b^2+\frac{2}{\sqrt{3}}a_5b_9\left(b\cdot p\right)+2a_5b_7\left(b\cdot p\right)+2a_5b_{14}\left(b\cdot p\right)^2\right]\omega_{\mu\nu}\Lambda_{\alpha\beta}\\
&+\left[a_5b_7\frac{\left(b\cdot p\right)^2}{p^2}-a_5b_6\left(b\cdot p\right)-2a_5b_{14}\left(b\cdot p\right)b^2-\frac{2}{\sqrt{3}}a_5b_8\left(b\cdot p\right)-2a_5b_{11}\frac{\left(b\cdot p\right)^2}{p^2}\right]\omega_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\\
&+\left[\frac{2}{3}a_5b_2\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_5b_9\left(b\cdot p\right)b^2-\frac{2}{\sqrt{3}}a_5b_8\left(b\cdot p\right)^2-\frac{2}{3}a_5b_3\left(b\cdot p\right)\right]\omega_{\mu\nu}\theta_{\alpha\beta}\\
&+\left[-\frac{1}{3}a_5b_2+\frac{1}{3}a_5b_3+a_5b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}\right]\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+\left[\frac{1}{\sqrt{3}}a_5b_5-a_5b_6\left(b\cdot p\right)+a_5b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{p^2}\right]\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\\
&+\left[\frac{2}{\sqrt{3}}a_5b_8p^2+\frac{2}{\sqrt{3}}a_5b_9\left(b\cdot p\right)\right]\Lambda_{\mu\nu}\theta_{\alpha\beta}+\left[a_5b_7+\frac{1}{\sqrt{3}}a_5b_9+a_5b_{10}\frac{b^2p^2-\left(b\cdot p\right)^2}{p^2}\right]\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\\
&+\left[2a_5b_{14}\left(b\cdot p\right)+2a_5b_{11}\right]\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}+\left[2a_5b_{12}\left(b\cdot p\right)+2a_5b_{13}p^2\right]\Lambda_{\mu\nu}\omega_{\alpha\beta}.
\end{aligned}
$$

$$
\begin{aligned}
a_6\tilde{\Pi}^{(2)}\mathcal{O}^{-1} = &\, a_6b_1\frac{\left(b\cdot p\right)}{p^2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta}+a_6b_2\left[\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)}{p^2}\Pi^{(1-a)}_{\mu\nu,\alpha\beta}-\frac{2}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta}+\frac{\left(b\cdot p\right)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}\right]\\
&+a_6b_3\left[\frac{2}{3}\Lambda_{\mu\nu}\theta_{\alpha\beta}-\frac{\left(b\cdot p\right)}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}\right]+\frac{1}{\sqrt{3}}a_6b_5\left[2\Lambda_{\mu\nu}\omega_{\alpha\beta}-\frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right]\\
&+a_6b_6\left[\left(b\cdot p\right)\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+b^2\Pi^{(1-a)}_{\mu\nu,\alpha\beta}+\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{1}{2}\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}-2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta}+\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right]\\
&+a_6b_7\left[b^2\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta}+2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)}{p^2}\tilde{\Pi}^{(\Lambda\Sigma)}_{\mu\nu,\alpha\beta}+\frac{\left(b\cdot p\right)^2}{2p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right]\\
&+\frac{1}{\sqrt{3}}a_6b_8\left[2\left(b\cdot p\right)\Lambda_{\mu\nu}\theta_{\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right]\\
&+\frac{1}{\sqrt{3}}a_6b_9\left[2b^2\Lambda_{\mu\nu}\theta_{\alpha\beta}-\frac{\left(b\cdot p\right)b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta}+2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\right]\\
&+a_6b_{10}\left[2b^2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\right]+a_6b_{11}\left[\frac{2\left(b\cdot p\right)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right]\\
&+a_6b_{12}\left[2b^2\Lambda_{\mu\nu}\omega_{\alpha\beta}-\frac{\left(b\cdot p\right)b^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right]+a_6b_{13}\left[2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right]\\
&+a_6b_{14}\left[2\left(b\cdot p\right)\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}-\frac{\left(b\cdot p\right)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}+2b^2\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}-\left(b\cdot p\right)b^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right],
\end{aligned}
$$

$$
\begin{aligned}
a_6\tilde{\Pi}^{(2)}\mathcal{O}^{-1} = &\left[a_6 b_2 + a_6 b_7 b^2 + a_6 b_6 \left(b\cdot p\right)\right]\tilde{\Pi}^{(2)}_{\mu\nu,\alpha\beta} + \left[a_6 b_1 \frac{(b\cdot p)}{p^2} - a_6 b_2 \frac{(b\cdot p)}{p^2} + a_6 b_6 b^2\right]\Pi^{(1-a)}_{\mu\nu,\alpha\beta} \\
&+ \left[a_6 b_7 \frac{(b\cdot p)^2}{2p^2} - \frac{1}{2} a_6 b_6 \left(b\cdot p\right) - a_6 b_{14} \left(b\cdot p\right) b^2 - \frac{1}{\sqrt{3}} a_6 b_8 \left(b\cdot p\right) - a_6 b_{11} \frac{(b\cdot p)^2}{p^2}\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \\
&+ \left[2a_6 b_7 + 2a_6 b_{10} b^2 + 2a_6 b_{14} \left(b\cdot p\right) + \frac{2}{\sqrt{3}} a_6 b_9\right]\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} \\
&+ \left[a_6 b_2 \frac{(b\cdot p)}{3p^2} - a_6 b_8 \frac{(b\cdot p)^2}{\sqrt{3}p^2} - a_6 b_3 \frac{(b\cdot p)}{3p^2} - a_6 b_9 \frac{(b\cdot p)\, b^2}{\sqrt{3}p^2}\right]\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} \\
&- \left[a_6 b_5 \frac{(b\cdot p)}{\sqrt{3}p^2} + a_6 b_{13} \frac{(b\cdot p)^2}{p^2} + a_6 b_{12} \frac{(b\cdot p)\, b^2}{p^2} - a_6 b_6 \frac{(b\cdot p)^2}{p^2}\right]\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta} \\
&+ \left[-a_6 b_7 \frac{(b\cdot p)}{p^2} - a_6 b_9 \frac{(b\cdot p)}{\sqrt{3}p^2} - a_6 b_{14} \frac{(b\cdot p)^2}{p^2} - a_6 b_{10} \frac{(b\cdot p)\, b^2}{p^2}\right]\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} \\
&+ \left[\frac{2}{\sqrt{3}} a_6 b_9 b^2 + \frac{2}{\sqrt{3}} a_6 b_8 \left(b\cdot p\right) - \frac{2}{3} a_6 b_2 + \frac{2}{3} a_6 b_3\right]\Lambda_{\mu\nu}\theta_{\alpha\beta} \\
&+ \left[\frac{2}{\sqrt{3}} a_6 b_5 + 2a_6 b_{12} b^2 + 2a_6 b_{13} \left(b\cdot p\right) - 2a_6 b_6 \left(b\cdot p\right)\right]\Lambda_{\mu\nu}\omega_{\alpha\beta} \\
&+ \left[a_6 b_6 + a_6 b_{11} \frac{2\left(b\cdot p\right)}{p^2} - a_6 b_7 \frac{(b\cdot p)}{p^2} + \frac{2}{\sqrt{3}} a_6 b_8 + 2a_6 b_{14} b^2\right]\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned}
$$

$$
\begin{aligned}
a_7\tilde{\Pi}^{(\theta\Sigma)}\mathcal{O}^{-1} = &\frac{1}{\sqrt{3}} a_7 b_1 \left[\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\left(b\cdot p\right)\theta_{\mu\nu}\omega_{\alpha\beta}\right] + \frac{1}{\sqrt{3}} a_7 b_3 \tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} \\
&+ \frac{2}{\sqrt{3}} a_7 b_4 \left(b\cdot p\right)\theta_{\mu\nu}\omega_{\alpha\beta} + a_7 b_5 \left[2\left(b\cdot p\right) P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right] \\
&+ \frac{1}{\sqrt{3}} a_7 b_6 \left[p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} - 2\left(b\cdot p\right)^2\theta_{\mu\nu}\omega_{\alpha\beta} + 2p^2\theta_{\mu\nu}\Lambda_{\alpha\beta} - 2\left(b\cdot p\right)\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right] \\
&+ \frac{1}{\sqrt{3}} a_7 b_7 \left[2\left(b\cdot p\right)\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)^2}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta} - \left(b\cdot p\right)\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right] \\
&+ a_7 b_8 \left\{2\left[b^2p^2 + \left(b\cdot p\right)^2\right] P^{(0-\theta)}_{\mu\nu,\alpha\beta} + p^2\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right\} \\
&+ a_7 b_9 \left[2\left(b\cdot p\right) b^2 P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{3p^2}\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} + \tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\right] \\
&+ \frac{1}{\sqrt{3}} a_7 b_{10} \left[2\left(b\cdot p\right) b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}\right] + \frac{2}{\sqrt{3}} a_7 b_{11} \frac{b^2p^2 + (b\cdot p)^2}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} \\
&+ \frac{1}{\sqrt{3}} a_7 b_{12} \left[2\left(b\cdot p\right) b^2\theta_{\mu\nu}\omega_{\alpha\beta} + 2\left(b\cdot p\right)\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2 - (b\cdot p)^2}{p^2}\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}\right] \\
&+ \frac{1}{\sqrt{3}} a_7 b_{13} \left\{2\left[b^2p^2 + \left(b\cdot p\right)^2\right]\theta_{\mu\nu}\omega_{\alpha\beta} + 2\left(b\cdot p\right)\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right\} \\
&+ \frac{1}{\sqrt{3}} a_7 b_{14} \left\{2\left[b^2p^2 + \left(b\cdot p\right)^2\right]\theta_{\mu\nu}\Lambda_{\alpha\beta} + 2\left(b\cdot p\right) b^2\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \left[b^2p^2 - \left(b\cdot p\right)^2\right]\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta}\right\},
\end{aligned}
$$

$$
\begin{aligned}
a_7\tilde{\Pi}^{(\theta\Sigma)}\mathcal{O}^{-1} = & \left\{2a_7b_5\left(b\cdot p\right)+2a_7b_9\left(b\cdot p\right)b^2+2a_7b_8\left[b^2p^2+\left(b\cdot p\right)^2\right]\right\}P^{(0-\theta)}_{\mu\nu,\alpha\beta} \\
& +\left\{a_7b_8p^2+\frac{1}{\sqrt{3}}a_7b_6p^2-\frac{1}{\sqrt{3}}a_7b_7\left(b\cdot p\right)+\frac{1}{\sqrt{3}}a_7b_{14}\left[b^2p^2-\left(b\cdot p\right)^2\right]\right\}\tilde{\Pi}^{(\omega\Lambda-a)}_{\mu\nu,\alpha\beta} \\
& +\left\{\frac{2}{\sqrt{3}}a_7b_4\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_7b_1\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{12}\left(b\cdot p\right)b^2\right. \\
& \left.-\frac{2}{\sqrt{3}}a_7b_6\left(b\cdot p\right)^2+\frac{2}{\sqrt{3}}a_7b_{13}\left[b^2p^2+\left(b\cdot p\right)^2\right]\right\}\theta_{\mu\nu}\omega_{\alpha\beta} \\
& +\left[\frac{1}{\sqrt{3}}a_7b_1+\frac{2}{\sqrt{3}}a_7b_{13}\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{14}\left(b\cdot p\right)b^2+2a_7b_{11}\frac{b^2p^2+\left(b\cdot p\right)^2}{\sqrt{3}p^2}-a_7b_7\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}\right]\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} \\
& +\left\{\frac{2}{\sqrt{3}}a_7b_6p^2+\frac{2}{\sqrt{3}}a_7b_7\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{12}\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{14}\left[b^2p^2+\left(b\cdot p\right)^2\right]+\frac{2}{\sqrt{3}}a_7b_{10}\left(b\cdot p\right)b^2\right\}\theta_{\mu\nu}\Lambda_{\alpha\beta} \\
& +\left[\frac{1}{\sqrt{3}}a_7b_3+a_7b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}\right]\tilde{\Sigma}_{\mu\nu}\theta_{\alpha\beta} \\
& +\left[a_7b_5+a_7b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}-\frac{2}{\sqrt{3}}a_7b_6\left(b\cdot p\right)\right]\tilde{\Sigma}_{\mu\nu}\omega_{\alpha\beta}+\left[a_7b_9+\frac{2}{\sqrt{3}}a_7b_7+a_7b_{10}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}\right]\tilde{\Sigma}_{\mu\nu}\Lambda_{\alpha\beta}.
\end{aligned}
$$

$$
\begin{aligned}
a_8\tilde{\Pi}^{(\theta\Lambda)}\mathcal{O}^{-1} = \frac{1}{\sqrt{3}}a_8b_1\left[\frac{(b\cdot p)}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\frac{(b\cdot p)^2}{p^2}\theta_{\mu\nu}\omega_{\alpha\beta}\right] \\
+\frac{1}{\sqrt{3}}a_8b_2\left[\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\theta_{\mu\nu}\omega_{\alpha\beta} - \frac{b^2p^2-(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta}\right] \\
+\frac{1}{\sqrt{3}}a_8b_3\left[\frac{b^2p^2-(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \Lambda_{\mu\nu}\theta_{\alpha\beta}\right] + a_8b_4\frac{(b\cdot p)^2}{\sqrt{3}p^2}\theta_{\mu\nu}\omega_{\alpha\beta} \\
+a_8b_5\left[\frac{(b\cdot p)^2}{p^2}P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\omega_{\alpha\beta} + \Lambda_{\mu\nu}\omega_{\alpha\beta}\right] \\
+\frac{1}{\sqrt{3}}a_8b_6\left[\frac{b^2p^2-(b\cdot p)^2}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2\left(b\cdot p\right)\theta_{\mu\nu}\Lambda_{\alpha\beta} - 2\left(b\cdot p\right)b^2\theta_{\mu\nu}\omega_{\alpha\beta} + \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} - 2\left(b\cdot p\right)\Lambda_{\mu\nu}\omega_{\alpha\beta}\right] \\
+\frac{1}{\sqrt{3}}a_8b_7\left[2b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} - \frac{(b\cdot p)\,b^2}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + 2\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta} - \frac{(b\cdot p)}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right] \\
+a_8b_8\left[2\left(b\cdot p\right)b^2P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{3p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right] \\
+a_8b_9\left[b^4P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\tilde{\Pi}^{(\theta\Lambda)}_{\mu\nu,\alpha\beta} + \tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}\right] + \frac{1}{\sqrt{3}}a_8b_{10}\left[b^4\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{p^2}\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}\right] \\
+2a_8b_{11}\frac{(b\cdot p)\,b^2}{\sqrt{3}p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{1}{\sqrt{3}}a_8b_{12}\left[b^4\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\theta_{\mu\nu}\Lambda_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\omega_{\alpha\beta}\right] \\
+\frac{1}{\sqrt{3}}a_8b_{13}\left[2\left(b\cdot p\right)b^2\theta_{\mu\nu}\omega_{\alpha\beta} + \frac{(b\cdot p)^2}{p^2}\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right] \\
+\frac{1}{\sqrt{3}}a_8b_{14}\left[2\left(b\cdot p\right)b^2\theta_{\mu\nu}\Lambda_{\alpha\beta} + b^4\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta} + \frac{b^2p^2-(b\cdot p)^2}{p^2}\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\right],
\end{aligned}
$$

$$
\begin{aligned}
a_8\tilde{\Pi}^{(\theta\Lambda)}\mathcal{O}^{-1} = &\left[a_8 b_9 b^4 + a_8 b_3 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2} + 2a_8 b_8 (b\cdot p)\, b^2\right.\\
&\left. + a_8 b_5 \frac{(b\cdot p)^2}{p^2} - a_8 b_2 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\right] P^{(0-\theta)}_{\mu\nu,\alpha\beta} + \left[a_8 b_9 + \frac{2}{\sqrt{3}} a_8 b_7 + a_8 b_{10}\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\right]\tilde{\Pi}^{(\Lambda\Lambda)}_{\mu\nu,\alpha\beta}\\
&+\left[\frac{1}{\sqrt{3}} a_8 b_{12} b^4 - 2a_8 b_1 \frac{(b\cdot p)^2}{\sqrt{3}p^2} + a_8 b_2 \frac{(b\cdot p)^2}{\sqrt{3}p^2} + a_8 b_4 \frac{(b\cdot p)^2}{\sqrt{3}p^2}\right.\\
&\left. + a_8 b_5 \frac{b^2p^2-(b\cdot p)^2}{3p^2} - \frac{2}{\sqrt{3}} a_8 b_6 (b\cdot p)\, b^2 + \frac{2}{\sqrt{3}} a_8 b_{13} (b\cdot p)\, b^2\right]\theta_{\mu\nu}\omega_{\alpha\beta}\\
&+\left[a_8 b_1 \frac{(b\cdot p)}{\sqrt{3}p^2} + a_8 b_6 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2} + 2a_8 b_{11}\frac{(b\cdot p)\, b^2}{\sqrt{3}p^2} + a_8 b_8 \frac{b^2p^2-(b\cdot p)^2}{3p^2}\right.\\
&\left. - a_8 b_7 \frac{(b\cdot p)\, b^2}{\sqrt{3}p^2} + \frac{1}{\sqrt{3}} a_8 b_{14} b^4 + a_8 b_{13}\frac{(b\cdot p)^2}{\sqrt{3}p^2} - a_8 b_2 \frac{(b\cdot p)}{\sqrt{3}p^2}\right]\theta_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}\\
&+\left[\frac{2}{\sqrt{3}} a_8 b_7 b^2 + \frac{1}{\sqrt{3}} a_8 b_2 + \frac{1}{\sqrt{3}} a_8 b_{10} b^4 + \frac{2}{\sqrt{3}} a_8 b_6 (b\cdot p)\right.\\
&\left. + \frac{2}{\sqrt{3}} a_8 b_{14} (b\cdot p)\, b^2 + a_8 b_{12}\frac{(b\cdot p)^2}{\sqrt{3}p^2} + a_8 b_9 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\right]\theta_{\mu\nu}\Lambda_{\alpha\beta}\\
&+\left[\frac{1}{\sqrt{3}} a_8 b_3 + a_8 b_9 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\right]\Lambda_{\mu\nu}\theta_{\alpha\beta} + \left[a_8 b_5 - \frac{2}{\sqrt{3}} a_8 b_6 (b\cdot p) + a_8 b_{12}\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\right]\Lambda_{\mu\nu}\omega_{\alpha\beta}\\
&+\left[a_8 b_8 + \frac{1}{\sqrt{3}} a_8 b_6 - a_8 b_7 \frac{(b\cdot p)}{\sqrt{3}p^2} + a_8 b_{14}\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\right]\Lambda_{\mu\nu}\tilde{\Sigma}_{\alpha\beta}.
\end{aligned}
$$

Assim, podemos escrever o sistema

$$
a_1 b_1 + a_1 b_6 (b\cdot p) - a_2 b_6 (b\cdot p) + a_5 b_1 (b\cdot p) - a_5 b_2 (b\cdot p) + a_5 b_6 b^2 p^2 = 1,
$$

$$
a_2 b_2 = 1,
$$

$$
\begin{aligned}
&- a_2 b_9 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2} + a_3 b_3 + a_3 b_9 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2} + a_4 b_5 + 2a_4 b_8 (b\cdot p) + a_4 b_9 \frac{(b\cdot p)^2}{p^2}\\
&+ 2a_7 b_5 (b\cdot p) + 2a_7 b_9 b^2 (b\cdot p) + 2a_7 b_8 \left[b^2p^2 + (b\cdot p)^2\right] + a_8 b_9 b^4 + a_8 b_3 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}\\
&+ 2a_8 b_8 b^2 (b\cdot p) + a_8 b_5 \frac{(b\cdot p)^2}{p^2} - a_8 b_2 \frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2} = 1,
\end{aligned}
$$

$$
\begin{aligned}
&- 2a_1 b_{13} (b\cdot p) - 2a_1 b_{12}\frac{(b\cdot p)^2}{p^2} + a_2 b_{12}\frac{(b\cdot p)^2}{p^2} + a_4 b_5 - \frac{2}{\sqrt{3}} a_4 b_6 (b\cdot p)\\
&+ a_4 b_{12}\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2} - \frac{2}{\sqrt{3}} a_5 b_5 (b\cdot p) + 2a_5 b_6 (b\cdot p)^2 - 2a_5 b_{12} b^2 (b\cdot p) - 2a_5 b_{13} (b\cdot p)^2 = 1,
\end{aligned}
$$

$$\frac{2}{3}a_2b_6\left(b\cdot p\right)-a_2b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}+a_3b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}+\frac{1}{\sqrt{3}}a_3b_5-\frac{2}{3}a_3b_6\left(b\cdot p\right)$$
$$+\frac{1}{\sqrt{3}}a_4b_3+a_4b_{12}\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+\frac{2}{\sqrt{3}}a_4b_{13}\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_7b_1\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_4\left(b\cdot p\right)$$
$$-\frac{2}{\sqrt{3}}a_7b_6\left(b\cdot p\right)^2+\frac{2}{\sqrt{3}}a_7b_{12}b^2\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{13}\left[b^2p^2+\left(b\cdot p\right)^2\right]$$
$$-2a_8b_1\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+a_8b_2\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+a_8b_4\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+a_8b_5\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}$$
$$-\frac{2}{\sqrt{3}}a_8b_6b^2\left(b\cdot p\right)+\frac{1}{\sqrt{3}}a_8b_{12}b^4+\frac{2}{\sqrt{3}}a_8b_{13}\left(b\cdot p\right)b^2=0,$$

$$-\frac{2}{\sqrt{3}}a_1b_8\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_1b_9\frac{\left(b\cdot p\right)^2}{p^2}+a_2b_9\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}$$
$$+\frac{1}{\sqrt{3}}a_4b_4+a_4b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}+\frac{2}{3}a_5b_2\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_5b_9b^2\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_5b_8\left(b\cdot p\right)^2-\frac{2}{3}a_5b_3\left(b\cdot p\right)=0,$$
$$a_2b_6+a_5b_1+a_5b_6\left(b\cdot p\right)+a_6b_1\frac{\left(b\cdot p\right)}{p^2}-a_6b_2\frac{\left(b\cdot p\right)}{p^2}+a_6b_6b^2=0,$$
$$a_1b_6+a_1b_7\frac{\left(b\cdot p\right)}{p^2}-a_2b_7\frac{\left(b\cdot p\right)}{p^2}+a_5b_2+a_5b_6\left(b\cdot p\right)+a_5b_7b^2=0,$$
$$a_2b_7+a_5b_6p^2+a_5b_7\left(b\cdot p\right)+a_6b_2+a_6b_6\left(b\cdot p\right)+a_6b_7b^2=0,$$

$$a_2b_7\frac{\left(b\cdot p\right)}{3p^2}-\frac{1}{3}a_2b_6+a_9b_1\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}-a_2b_{14}\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}+\frac{1}{\sqrt{3}}a_3b_8+\frac{1}{3}a_3b_6+a_3b_{14}\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}-a_3b_7\frac{\left(b\cdot p\right)}{3p^2}$$
$$+\frac{1}{\sqrt{3}}a_4b_{13}+\frac{2}{\sqrt{3}}a_4b_{11}\frac{\left(b\cdot p\right)}{p^2}+a_4b_{14}\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+\frac{1}{\sqrt{3}}a_7b_1-a_7b_7\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+2a_7b_{11}\frac{b^2p^2+\left(b\cdot p\right)^2}{\sqrt{3}p^2}$$
$$+\frac{2}{\sqrt{3}}a_7b_{13}\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{14}\left(b\cdot p\right)b^2-a_8b_2\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}+a_8b_6\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}$$
$$+a_8b_8\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}-a_8b_7\frac{\left(b\cdot p\right)b^2}{\sqrt{3}p^2}+2a_8b_{11}\frac{\left(b\cdot p\right)b^2}{\sqrt{3}p^2}+a_8b_{13}\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+\frac{1}{\sqrt{3}}a_8b_{14}b^4=0,$$

$$\frac{1}{\sqrt{3}}a_1b_8+a_1b_9\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}-a_2b_9\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}-\frac{1}{3}a_5b_2+\frac{1}{3}a_5b_3+a_5b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}$$
$$+a_6b_2\frac{\left(b\cdot p\right)}{3p^2}-a_6b_3\frac{\left(b\cdot p\right)}{3p^2}-a_6b_8\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}-a_6b_9\frac{\left(b\cdot p\right)b^2}{\sqrt{3}p^2}+\frac{1}{\sqrt{3}}a_7b_3+a_7b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}=0,$$

$$-a_2b_{10}\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}-\frac{2}{3}a_2b_7+\frac{2}{3}a_3b_7+\frac{1}{\sqrt{3}}a_3b_9+a_3b_{10}\frac{b^2p^2-\left(b\cdot p\right)^2}{3p^2}$$
$$+a_4b_{10}\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+\frac{1}{\sqrt{3}}a_4b_{12}+\frac{2}{\sqrt{3}}a_4b_{14}\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_6p^2+\frac{2}{\sqrt{3}}a_7b_7\left(b\cdot p\right)$$
$$+\frac{2}{\sqrt{3}}a_7b_{10}\left(b\cdot p\right)b^2+\frac{2}{\sqrt{3}}a_7b_{12}\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_7b_{14}\left[b^2p^2+\left(b\cdot p\right)^2\right]+\frac{1}{\sqrt{3}}a_8b_2$$
$$+\frac{2}{\sqrt{3}}a_8b_7b^2+\frac{1}{\sqrt{3}}a_8b_{10}b^4+a_8b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}+a_8b_{12}\frac{\left(b\cdot p\right)^2}{\sqrt{3}p^2}+\frac{2}{\sqrt{3}}a_8b_{14}\left(b\cdot p\right)b^2=0,$$

$$\frac{1}{\sqrt{3}}a_2b_9+\frac{2}{\sqrt{3}}a_5b_8p^2+\frac{2}{\sqrt{3}}a_5b_9\left(b\cdot p\right)-\frac{2}{3}a_6b_2+\frac{2}{3}a_6b_3+\frac{2}{\sqrt{3}}a_6b_8\left(b\cdot p\right)$$
$$+\frac{2}{\sqrt{3}}a_6b_9b^2+\frac{1}{\sqrt{3}}a_8b_3+\frac{2}{\sqrt{3}}a_8b_6\left(b\cdot p\right)+a_8b_9\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}=0,$$

$$a_2b_{10}+2a_5b_{10}\left(b\cdot p\right)+2a_5b_{14}p^2+2a_6b_7+2a_6b_{10}b^2+2a_6b_{14}\left(b\cdot p\right)+\frac{2}{\sqrt{3}}a_6b_9$$
$$+a_8b_9+\frac{2}{\sqrt{3}}a_8b_7+a_8b_{10}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}=0,$$

$$a_1b_{14}\left(b\cdot p\right)+a_1b_{11}-a_2b_{14}\left(b\cdot p\right)+\frac{1}{2}a_5b_6p^2-\frac{1}{2}a_5b_7\left(b\cdot p\right)+\frac{1}{\sqrt{3}}a_5b_8p^2+a_5b_{14}\left[b^2p^2-\left(b\cdot p\right)^2\right]$$
$$-\frac{1}{2}a_6b_6\left(b\cdot p\right)+a_6b_7\frac{\left(b\cdot p\right)^2}{2p^2}-\frac{1}{\sqrt{3}}a_6b_8\left(b\cdot p\right)-a_6b_{11}\frac{\left(b\cdot p\right)^2}{p^2}-a_6b_{14}\left(b\cdot p\right)b^2$$
$$+\frac{1}{\sqrt{3}}a_7b_6p^2-\frac{1}{\sqrt{3}}a_7b_7\left(b\cdot p\right)+a_7b_8p^2+\frac{1}{\sqrt{3}}a_7b_{14}\left[b^2p^2-\left(b\cdot p\right)^2\right]=0,$$

$$-2a_1b_{10}\frac{\left(b\cdot p\right)^2}{p^2}-2a_1b_{14}\left(b\cdot p\right)+a_2b_{10}\frac{\left(b\cdot p\right)^2}{p^2}+\frac{2}{\sqrt{3}}a_4b_7+a_4b_9$$
$$+a_4b_{10}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}-2a_5b_7\left(b\cdot p\right)-\frac{2}{\sqrt{3}}a_5b_9\left(b\cdot p\right)-2a_5b_{10}\left(b\cdot p\right)b^2-2a_5b_{14}\left(b\cdot p\right)^2=0,$$

$$a_2b_{12}+2a_5b_{12}\left(b\cdot p\right)+2a_5b_{13}p^2+\frac{2}{\sqrt{3}}a_6b_5-2a_6b_6\left(b\cdot p\right)+2a_6b_{12}b^2$$
$$+2a_6b_{13}\left(b\cdot p\right)+a_8b_5-\frac{2}{\sqrt{3}}a_8b_6\left(b\cdot p\right)+a_8b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}=0,$$

$$-2a_1b_{11}\frac{\left(b\cdot p\right)}{p^2}-2a_1b_{14}\frac{\left(b\cdot p\right)^2}{p^2}+a_2b_{14}\frac{\left(b\cdot p\right)^2}{p^2}$$
$$+\frac{1}{\sqrt{3}}a_4b_6-a_4b_7\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}+a_4b_8+a_4b_{14}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}-a_5b_6\left(b\cdot p\right)$$
$$+a_5b_7\frac{\left(b\cdot p\right)^2}{p^2}-\frac{2}{\sqrt{3}}a_5b_8\left(b\cdot p\right)-2a_5b_{11}\frac{\left(b\cdot p\right)^2}{p^2}-2a_5b_{14}\left(b\cdot p\right)b^2=0,$$

$$a_1b_{12}\frac{\left(b\cdot p\right)}{p^2}+a_1b_{13}-a_2b_{12}\frac{\left(b\cdot p\right)}{p^2}+\frac{1}{\sqrt{3}}a_5b_5-a_5b_6\left(b\cdot p\right)+a_5b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{p^2}-a_6b_5\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}$$
$$+a_6b_6\frac{\left(b\cdot p\right)^2}{p^2}-a_6b_{12}\frac{\left(b\cdot p\right)b^2}{p^2}-a_6b_{13}\frac{\left(b\cdot p\right)^2}{p^2}+a_7b_5-\frac{2}{\sqrt{3}}a_7b_6\left(b\cdot p\right)+a_7b_{12}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}=0,$$

$$a_2b_{14}+2a_5b_{14}\left(b\cdot p\right)+2a_5b_{11}+a_6b_6-a_6b_7\frac{\left(b\cdot p\right)}{p^2}+a_6b_{11}\frac{2\left(b\cdot p\right)}{p^2}+\frac{2}{\sqrt{3}}a_6b_8+2a_6b_{14}b^2$$
$$+\frac{1}{\sqrt{3}}a_8b_6+a_8b_8-a_8b_7\frac{\left(b\cdot p\right)}{\sqrt{3}p^2}+a_8b_{14}\frac{b^2p^2-\left(b\cdot p\right)^2}{\sqrt{3}p^2}=0,$$

$$a_1 b_{10}\frac{(b\cdot p)}{p^2}+a_1 b_{14}-a_2 b_{10}\frac{(b\cdot p)}{p^2}+a_5 b_7+\frac{1}{\sqrt{3}}a_5 b_9+a_5 b_{10}\frac{b^2p^2-(b\cdot p)^2}{p^2}-a_6 b_7\frac{(b\cdot p)}{p^2}$$
$$-a_6 b_9\frac{(b\cdot p)}{\sqrt{3}p^2}-a_6 b_{10}\frac{(b\cdot p)\,b^2}{p^2}-a_6 b_{14}\frac{(b\cdot p)^2}{p^2}+a_7 b_9+\frac{2}{\sqrt{3}}a_7 b_7+a_7 b_{10}\frac{b^2p^2-(b\cdot p)^2}{\sqrt{3}p^2}=0.$$